Team Decision Theory and Information Structures

August 9, 2015

"The theme of this week's team meeting is, 'Take It Right to the Edge'."

References

Yu-Chi Ho, ”Team Decision Theory and Information Structures”, Proceedings of IEEE, Vol. 68, No. 6, June 1980

Outline

- Introduction
- General mathematical model
- Example and variations
- Signaling and information theory
- Conclusions

Example

Sun ?

Bangalore

Mr. B

Shine in Electronic City

Mr. B \ Mr. H	GO	Don't GO
GO	10	-3
Don't GO	-3	0

Meeting next day if no rain

Electronic City

RAIN?

Rain in Electronic City

Mr. B \ Mr. H	GO	Don't GO
GO	-4	-2
Don't GO	-2	5

Hosur

Mr. H

Solution

ξ_B	r	r	r	r	s	s	s	s
ξ_H	r	r	s	s	r	r	s	s
ξ_E	r	s	r	s	r	s	r	s
Prob.	0.25	0.05	0.1	0.1	0.1	0.1	0.05	0.25

Expected payoff

$$\bar{J} = \sum_{\xi} L(u_B, u_H, u_E)\text{Pr}(\xi_B, \xi_H, \xi_E)$$

where $u_B = \gamma(\xi_B)$ and $u_H = \gamma(\xi_H)$ and L is the payoff function

What is the optimal *decision rule*?

Main Ingredients

- Each decision maker has access to *different but correlated information* about some underlying uncertainty

- *Need for coordinated actions* on the part of all decision makers in order to realize the payoff

Note 1: In absence of any of the above problem simplifies

Note 2: Any kind of communication is permitted *beforehand*

Formal Model

- States of nature: $\xi = [\xi_1, \xi_2, \cdot, \xi_m]$ with given distribution $p(\xi)$
- Set of observations: $z = [z_1, \cdots, z_n]$, where for all i

$$z_i = \eta_i(\xi_1, \cdots, \xi_m)$$

$\{\eta_i | i = 1, \cdots, n\}$: Information structure of the problem

- Set of decision variables: $u = [u_1, \cdots, u_n]$
- Strategy (decision rule): $\gamma_i : Z_i \rightarrow U_i$ where for all i $\gamma_i \in \Gamma_i$ and $u_1 = \gamma(z_i)$
- Loss (payoff) function: $L : \Xi \;\; x \;\; U \rightarrow R$ i.e.,

$$Loss = L(u_1, \cdots, u_n, \xi_1, \cdots, \xi_m)$$

$$\boxed{\text{minimize}_{\{\gamma_1, \cdots, \gamma_n\} \in \Gamma_1 \;\; x \cdots \;\; x \;\; \Gamma_n\}} J = \mathbb{E}_\xi[L(u = \gamma(\eta(\xi)), \xi)]}$$

Design Issues

- What should one do?: Design of decision rules γ_i
- What should know what?: Design of information structure η

i^{th} Decision Maker's View Point

$\bar{\gamma}_i$: strategy of all other DMs (fixed) and DM$_i$ knows it.

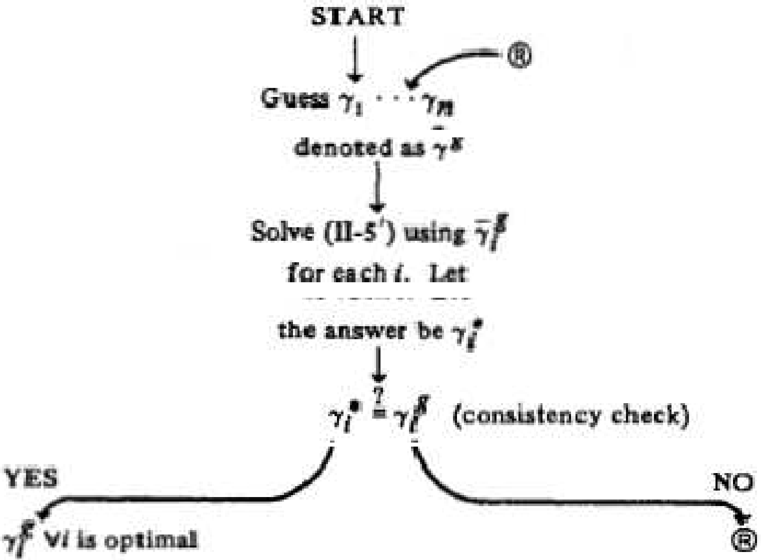
DM$_i$'s problem

$$\begin{aligned}
\min_{\gamma_i \in \Gamma_i} J(\gamma_i, \bar{\gamma}_i) &= \mathbb{E}_\xi[L(u_i = \gamma_i(\eta_i(\xi)), \bar{\gamma}_i, \xi)] \\
&= \min_{\gamma_i \in \Gamma_i} \mathbb{E}_{z_i}\mathbb{E}_{\xi|z_i}[L(\gamma_i, \bar{\gamma}_i, \xi)] \\
&= \mathbb{E}_{z_i} \min_{u_i \in U_i} \mathbb{E}_{\xi|z_i}[L(u_i, \bar{\gamma}_i, \xi)]
\end{aligned}$$

Hence, for all i

$$\boxed{\min_{u_i \in U_i} \mathbb{E}_{z_i} \min_{u_i \in U_i} \mathbb{E}_{\xi|z_i}[L(u_i, \bar{\gamma}_i, \xi)] \equiv \min_{u_i \in U_i} J_i(u_i, z_i, \bar{\gamma}_i)}$$

A Procedure


START
®
Guess $\gamma_1 \cdots \gamma_m$
denoted as $\bar{\gamma}^g$
Solve (II-5') using $\bar{\gamma}_i^g$
for each i. Let
the answer be γ_i^*
$\gamma_i^* \stackrel{?}{=} \gamma_i^g$ (consistency check)
YES
NO
γ_i^g $\forall i$ is optimal
®

Examples: Two Person Team Problems

- Loss

$$L = \frac{1}{2}(x + au_1 + u_2)^2 + hu_1^2 + gu_2^2, \;\; a, g \geq 0 \text{ and } h > 0$$

- Observations

$$y_1 = x + bv_1 \quad b > 0$$

$$y_1 = x + cu_1 + dv_2 \quad c \geq 0, d > 0$$

where $x \sim \mathcal{N}(0, \sigma^2)$, $v_1 \sim \mathcal{N}(0, 1)$ and $v_2 \sim \mathcal{N}(0, 1)$

- Difference?

First Variation

- Loss

$$L = \frac{1}{2}(x + u_1 + u_2)^2 + \frac{1}{2}u_1^2 + \frac{1}{2}u_2^2, \;\; a = 1, h = g = \frac{1}{2}$$

- Observations

$$y_1 = x + v_1 \quad b = 1$$

$$y_1 = x + v_2 \quad c = 0, d = 1$$

where $x \sim \mathcal{N}(0, \sigma^2)$, $v_1 \sim \mathcal{N}(0, 1)$ and $v_2 \sim \mathcal{N}(0, 1)$

- Interpretation!

Solution

Guess $u_1 = k_1 z_1$ and $u_2 = k_2 z_2$ for 'procedure'

$$u_1 = -\frac{1}{2}\mathbb{E}_{\xi|z_1}(k_2 z_2 + x)$$

$$u_2 = -\frac{1}{2}\mathbb{E}_{\xi|z_2}(k_1 z_1 + x)$$

Consistency condition gives

$$\begin{bmatrix} 1 & \frac{\sigma^2}{2(\sigma^2+1)} \\ \frac{\sigma^2}{2(\sigma^2+1)} & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\sigma^2}{2(\sigma^2+1)}$$

Thus, $k_1^* = k_2^* = -\frac{\sigma^2}{3\sigma^2+2)}$ (Global optimal?)

Proof: Global optimal

Let $u_i^* = \gamma_i * (z_i) = k_i * z_i$ for $i = 1, 2$ denote the *'individually'* optimal solution; and $u_i = \gamma_i(z_i)$ be any other strategy.

Using strict convexity of L

$$L(u_1, u_2, \xi) > L(u_1^*, u_2^*, \xi) + \sum_{i=1}^{2} \frac{\partial L}{\partial u_i}\bigg|_{u_1^*, u_2*} (u_i - u_i^*)$$

Now take expectation on both sides and substituting $u_i \to \gamma_i$ and $u_i^* \to \gamma_i^*$

Proof Contd.

$$
\begin{aligned}
J(\gamma_1,\gamma_2) &\equiv \mathbb{E}[u_1=\gamma(z_1),u_2=\gamma(z_2),\xi] \\
&> J(\gamma_1^*,\gamma_2^*)+\mathbb{E}\left\{\sum_{i=1}^{2}\frac{\partial L}{\partial u_i}\bigg|_{u_1^*,u_2*}(\gamma_i-\gamma_i^*)\right\} \\
&= J(\gamma_1^*,\gamma_2^*)+\mathbb{E}_{z_i}\left[\sum_{i=1}^{2}\mathbb{E}_{\xi|z_i}\left\{\frac{\partial L}{\partial u_i}\bigg|_{u_1^*,u_2*}\right\}(\gamma_i-\gamma_i^*)\right] \\
&= J(\gamma_1,\gamma_2)
\end{aligned}
$$

Proposition

In linear-Quadratic Gaussian team with Q¿0

$$L=\frac{1}{2}u^TQu+u^TS\xi,\ \text{when } Q>0,\ \text{and } \xi\sim\mathcal{N}(0,\Sigma)$$

and

$$y=H\xi$$

The unique optimal solution is linear in the information and is

"Well, Ladies and Gentlemen, I'm sure my little talk has made you all think."