Group Discussion
Compressed Sensing for Quaternion Signal

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Organization

- Introduction to Quaternion
- Quaternion Hilbert Space
- Least square problem
- Compressed sensing for Quaternion Signals

Quaternion

- Quaternions are generally represented in the form:

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\}$$

- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

$$\begin{bmatrix} x & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{i} & \mathbf{i} & -1 & \mathbf{k} & -\mathbf{j} \\ \mathbf{j} & \mathbf{j} & -\mathbf{k} & -1 & \mathbf{i} \\ \mathbf{k} & \mathbf{k} & \mathbf{j} & -\mathbf{i} & -1 \end{bmatrix}$$

Quaternion ring

- $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$ and $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \in \mathbb{H}$
  $a + b = (a_0 + b_0) + (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k} \in \mathbb{H}$
  where addition of the real components $a_i + b_i$ is the usual addition in $\mathbb{R}$.
- addition in $\mathbb{H}$ is associative - it is easy to show that $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{H}$, using the fact that $(a_i + b_i) + c_i = a_i + (b_i + c_i)$ for all $a_i, b_i, c_i \in R$,
- $\mathbb{H}$ has an additive identity, namely the real number $0 = 0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$,
- every element of $\mathbb{H}$ has an additive inverse - if $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$ then $-a = (-a_0) + (-a_1)\mathbf{i} + (-a_2)\mathbf{j} + (-a_3)\mathbf{k}$ is another quaternion (as all elements of $\mathbb{R}$ have negatives in $\mathbb{R}$) and $aa = 0$,
- since addition is commutative in $\mathbb{R}$, it is also commutative in $\mathbb{H}$.
- This all shows that the quaternions form an abelian group wrt addition.

Multiplication

- $ab = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)\mathbf{i} + (a_0b_2 + a_0b_0 + a_1b_3 - a_3b_1)\mathbf{j} + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)\mathbf{k}$
- multiplication in H is associative, which follows from the fact that the assoc and distributive laws hold in $\mathbb{R}$,
- $\mathbb{H}$ has a multiplicative identity, namely the real number $1 = 1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$,
- the left and right distributive laws hold in $\mathbb{H}$, which follows from the fact that the associative and distributive laws hold in $\mathbb{R}$.
- The real quaternions form a unital ring wrt addition and multiplication as defined above.

Division ring

Definition

$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$, define

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Definition

$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$, define

$$\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$

- $q\overline{q} = |q|^2$
- $q^{-1} = \frac{\overline{q}}{|q|^2}$, where $q \neq 0$
- The ring of real quaternions is a division ring.
  (Recall that a division ring is a unital ring in which every element has a multiplicative inverse. It is not necessarily also a commutative ring. A division ring that is commutative is simply a field.)

Quaternion Hilbert Space

Definition

Let H be a right $\mathbb{H}$- module. A map

$$\langle .|. \rangle : H \times H \to \mathbb{H}$$

satisfying:

- If $u \in H$, then $\langle u|u \rangle = 0 \implies u = 0$
- $\langle u|v + w \cdot q \rangle = \langle u|v \rangle + \langle u|w \rangle \cdot q$, for all $u, v \in H$ and $q \in \mathbb{H}$
- $\langle u|v \rangle = \overline{\langle v|u \rangle}$, for all $u, v \in H$,

is called an inner product on H. If we define $\|u\|^2 = \langle u|u \rangle$, for all $u \in H$, then $\|\|$ is a norm on H and is called the norm induced by $\langle .|. \rangle$. If $(H, \|\|)$ is complete space then it is called a right quaternionic Hilbert space.

Quaternion to complex

- $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H} = (q_0 + q_1\mathbf{i}) + (q_2 + q_3\mathbf{i}) \cdot \mathbf{j}$.
- Let H be seperable right quaternionic Hilbert space (If H has a countable dense subset then H is called seperable.)
- Let $A \in \mathbb{M}_{m \times M}(\mathbb{H})$ with $m \leq M$ be a frame and $Ax = y$ be a quaternion linear system.
- $A = A_1 + A_2 \cdot \mathbf{j}$, where $A_1, A_2 \in \mathbb{C}^{m \times M}$, $x = x_1 + x_2 \cdot \mathbf{j} \in \mathbb{H}^M$, where $x_1, x_2 \in \mathbb{C}^M$, and $y = y_1 + y_2 \cdot \mathbf{j} \in \mathbb{H}^m$, where $y_1, y_2 \in \mathbb{C}^m$,
- $Ax = y$ is equivalent to

$$\chi_A \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -\overline{y}_2 \end{bmatrix},$$

where $\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{bmatrix}$

Optimization problem

Definition

Let $x = (x_{1i} + x_{2i} \cdot \mathbf{j})_{i=1}^{M} \in \mathbb{H}^M$. Define

$$\|x\|_p = \left(\sum_{i=1}^{M} |(x_{1i} + x_{2i} \cdot \mathbf{j})|^p \right)^{\frac{1}{p}}, \quad p = 1, 2.$$

$$P_p : \min_{x} \|x\|_p \text{ subject to } \quad Ax = y.$$

$$P_p : \min_{\begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix}} \left\| \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix} \right\|_p \text{ subject to } \quad \chi_A \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -\overline{y}_2 \end{bmatrix}.$$

If A^+ be such that $AA^+(y_1 + y_2 \cdot \mathbf{j}) = y_1 + y_2 \cdot \mathbf{j}$ then
$\chi_A \chi_{A^+} \begin{pmatrix} y_1 \\ -\bar{y}_2 \end{pmatrix} = \chi_{AA^+} \begin{pmatrix} y_1 \\ -\bar{y}_2 \end{pmatrix} = AA^+(y_1 + y_2 \cdot \mathbf{j}) = y_1 + y_2 \cdot \mathbf{j}.$
$\implies (\chi_A)^+ = \chi_{A^+}$
we are using complex result to extend it to quaternion case.

Compressed sensing for quaternion signals

$$P_0 : \min_x \|x\|_0 \text{ subject to } \quad Ax = y.$$

$$P_1 : \min_x \|x\|_1 \text{ subject to } \quad Ax = y.$$

equivalent to

$$P_1 : \min_{\begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix}} \| \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix} \|_1 \text{ subject to } \quad \chi_A \begin{bmatrix} x_1 \\ -\overline{x}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -\overline{y}_2 \end{bmatrix}.$$

- If A satisfies $(2k, \delta)$ RIP then P_1 provides P_0 solution.

- $\delta_k \in (0,1)$, $k-$ sparse signals $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$(1-\delta_k)\,\|X\|_{\mathbb{C}}^2 \;\leq\; \|\chi_A X\|^2 \leq (1+\delta_k)\,\|X\|_{\mathbb{C}}^2$$

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$$\|\chi_A \begin{pmatrix} x_1 \\ -\overline{x}_2 \end{pmatrix}\| = \|A(x_1 + x_2 \cdot \mathbf{j})\|, \text{ for all } \begin{pmatrix} x_1 \\ -\overline{x}_2 \end{pmatrix} \in \mathbb{C}^n \oplus \mathbb{C}^n.$$

$\Longrightarrow$

$$\|\chi_A\| = \|A\|.$$

- $\|x_1 + x_2 \cdot \mathbf{j}\| = \| \begin{pmatrix} x_1 \\ -\overline{x}_2 \end{pmatrix} \|$

$$(1-\delta_k)\,\|x_1 - \overline{x}_2 \cdot \mathbf{j}\|_{\mathbb{Q}}^2 \;\leq\; \|A(x_1 - \overline{x}_2 \cdot \mathbf{j})\|^2 \leq (1+\delta_k)\,\|x_1 - \overline{x}_2 \cdot \mathbf{j}\|_{\mathbb{Q}}^2$$

- χ_A - $(2k,\delta)-$RIP $\Longrightarrow$ A - $(k,\delta)-$RIP
- A - $(k,\delta)-$RIP $\Longrightarrow$ χ_A - $(k,\delta)-$RIP

- In literature l_1 minimization problem is solved through second-order cone programming.
- One can approach through solving the corresponding complex system of equations.
- OMP for quaternions

References

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Thank You