Continuous Time Markov Chains

An Introduction

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Applications

- Queuing Theory
- Can be seen as generalizations of DTMCs and Poisson processes
- Also used for modelling biological systems

Formal Definition

A continuous time stochastic process $\{X(t), t \geq 0, X(t) \in \mathbb{Z}^{+}\}$ is known as a continuous time Markov Chain in for all $s, t \geq 0$ and for all non negative integers $i, j, x(u)\ (0 \leq u \leq s)$

$$P\{X(t+s) = j | X(s) = i, X(u) = x(u)\}$$
$$= P\{X(t+s) = j | X(s) = i\}$$

Stationarity is achieved when the above is independent of s

Can something be said about the distribution time in a state?

Properties

The time spent in a state i before transitioning into the next state is exponentially distributed with mean $\frac{1}{v_i}$

The transition probability P_{ij} of transitioning into state j from state i satisfies

$$P_{ii} = 0$$
$$\sum_j P_{ij} = 1$$

The next state is independent of the time spent in the present state. (Why?)

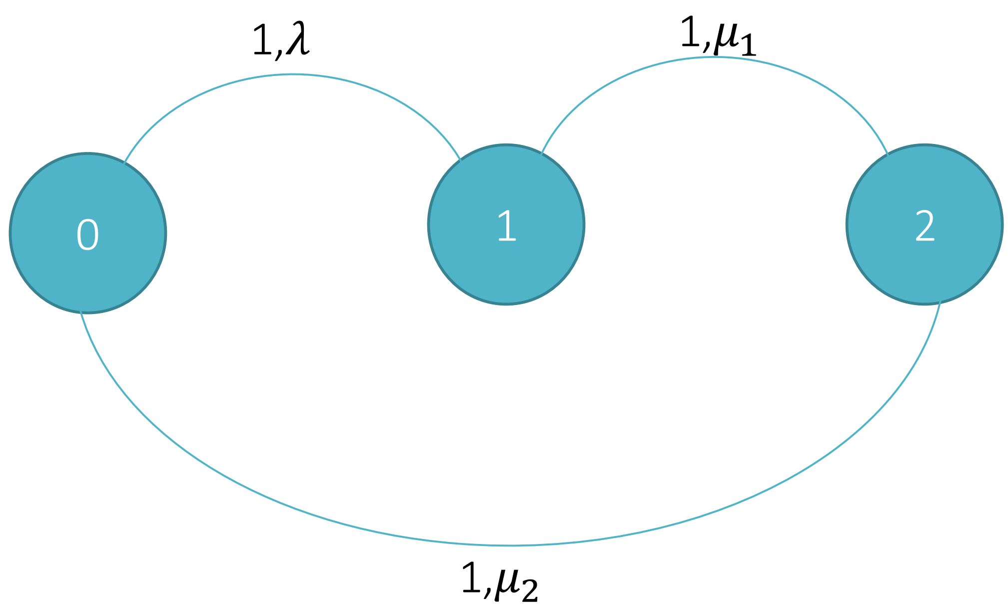
Example

Consider a shoe shine shop with the following rules

a. *There are two chairs with service rates* μ_1 *and* μ_2

b. *A new customer enterswith a rate* λ, *only when the system is vacant*

The occupied chair may be used as the state of the system



Birth and Death Processes

System state represented by number of people in the system

For state i

a. Arrival rate = λ_i
b. Departure rate = μ_i
c. Transition rate $v_i = \lambda_i + \mu_i$ (How?)
d. $P_{i,i+1} = P\{Arrival\ before\ departure\} = \frac{\lambda_i}{\lambda_i+\mu_i}$
e. $P_{i,i-1} = P\{Departure\ before\ arrival\} = \frac{\mu_i}{\lambda_i+\mu_i}$

Some Examples

1. The Yule Process
2. The Poisson Process
3. Linear Growth with immigration
4. Queueing systems with one or more servers

State-Transition Time

Let T_i denote the time that the system takes to enter state $i + 1$ starting from state i

Define

$$I_i = \begin{cases} 1 & \textit{The system transtions to } i+1 \\ 0 & \textit{The system transtions to } i-1 \end{cases}$$

$$E[T_i | I_i = 1] = \frac{1}{\lambda_i + \mu_i}$$

Mean Time for Transition

$$E[T_i | I_i = 0] = \frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i]$$

Mean time for transition +mean time to enter $i + 1$ from $i - 1$

State Transition Time

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

This may be computed recursively with

$$E[T_0] = \frac{1}{\lambda_0}$$

Mean time required for getting from state i to j

$$\sum_{k=i}^{j} E[T_k]$$

Example : The M/M/1 queue

State Transition Time

$$E[T_i|I_i] = \frac{1}{\lambda_i + \mu_i} + (1 - I_i)(E[T_{i-1}] + E[T_i])$$

But

$$var(X) = E[var(X|Y)] + var(E[X|Y])$$

Using this,

$$var(E[T_i|I_i]) = var(I_i)(E[T_{i-1}] + E[T_i])^2$$

$$var(T_i|I_i = 1) = \frac{1}{(\lambda_i + \mu_i)^2}$$

Variance of Transition time

$$var(T_i|I_i = 0) = \frac{1}{(\lambda_i + \mu_i)^2} + var(T_i) + var(T_{i-1})$$

State Transition Time

$$var(T_i) = \frac{1}{\lambda_i(\lambda_i + \mu_i)} + \frac{\mu_i}{\lambda_i} var(T_{i-1}) + \frac{\mu_i}{\lambda_i + \mu_i} (E[T_i] + E[T_{i-1}])^2$$

This may be computed recursively by using

$$var(T_0) = \frac{1}{\lambda_0^2}$$

$$var(time\ to\ go\ from\ i\ to\ j) = \sum_{k=i}^{j} var(T_i)$$

Transition Probabilities

Let

$$P_{ij}(t) = P\{X(t+s) = j | X(s) = i\}$$

and

$$q_{ij} = v_i P_{ij}$$

consequently,

$$v_i = \sum_j q_{ij}$$

It may be shown that

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{jk}(t)$$

and

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$$

$$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}$$

Kolmogrov’s Equations

Backward

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

Forward

$$P'_{ij}(t) = \sum_{k \neq i} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$$

Conclusions

CTMCs may be used to model queueing systems

The mean time spent in a state may be computed recursively