Coverage Analysis of Cellular Networks with Imperfect Channel Knowledge

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Outline

- Preliminaries
  - Poisson Point Process (PPP)
- System Model
  - Uplink Training
  - Downlink Transmission
- Problem Statement
- Derivations
- Results

Poisson Point Process (PPP)

- Spatial generalisation of a Poisson Process
- **Stationary**: Defined over complete plane $\mathbb{R}^2$
- **Definition**
  1. The number of points in two bounded and disjoint sets $A_1 \in \mathbb{R}^2$ and $A_2 \in \mathbb{R}^2$ are independent
  2. The number of points in a bounded set $A \subset \mathbb{R}^2$ are poisson distributed with mean $\lambda|A|$

$$\mathbb{P}(\phi(A) = n) = exp(-\lambda|A|)\frac{(\lambda|A|)^n}{n!}$$

where λ is the density of point process ϕ and $|A|$ represents the area of the bounded region

Properties of PPP

- **First Contact Distance Distribution**
  - **Probability Distribution Function (PDF)**

$$f_R(r) = 2\pi r\lambda exp(-\pi r^2\lambda)$$

- **Thinning of PPP of density λ**
  - Select a point with probability p independently
  - Results in two independent PPPs of density $p\lambda$ and $(1-p)\lambda$
- **Slivnyak's Theorem**

$$\mathbb{P}^{!o} = \mathbb{P}$$

Reduced palm distribution is same as PPP distribution itself.

Theorem

- **Campbell's Theorem: Sums over PPP**
  - If ϕ is a PPP of density λ and $f(x) : \mathbb{R}^2 \to \mathbb{R}^+$, then

$$\mathbb{E}[\sum_{x \in \phi} f(x)] = \lambda \int_{\mathbb{R}^2} f(x)dx$$

- **Probability Generating Functional (PGFL): Products over PPP**
  - If ϕ is a PPP of density λ and $f(x) : \mathbb{R}^2 \to [0,1]$, then

$$\mathbb{E}[\prod_{x \in \phi} f(x)] = exp\left(-\lambda \int_{\mathbb{R}^2} (1 - f(x))dx\right)$$

System Model

- Time Division Duplex (TDD) system
- **Training**
  - Uplink Training: Training duration is L_τ symbol interval
  - P_τ, Training Power per Symbol
  - L_τ set of orthogonal sequences
  - Randomly choose sequence
- **Downlink**
  - Base Stations (BS): PPP ϕ_B of density λ_B
  - Mobile Users (MU): Independent PPP ϕ_m of density λ_m
  - MU connects to the nearest BS
  - BS transmits with power P_D
  - Rayleigh Fading
  - Path Loss Model: $min\{1, r^{-\alpha}\}$

Problem Statement

- **Coverage Probability**: The probability that a randomly selected user can achieve a target $SINR$, say θ

$$P_c = \mathbb{P}(SINR > \theta)$$

Outage Probability: Randomly selected user is in outage

$$P_o = 1 - P_c = \mathbb{P}(SINR < \theta)$$

- **Ergodic Capacity**: Average rate achieved by typical user

$$C = \frac{(L - L_\tau)}{L}\mathbb{E}[\ln(1 + SINR)]$$

where L is the Coherence interval

Goal

- To Compute: Coverage probability $P_{c|r}$ conditioned over first contact distance
- Uncondition by using PDF of first contact distance (Previous Slides: First Contact Distribution)
- Study the effect of training duration and $SINR$ threshold θ on the coverage probability
- Comparison with the perfect channel state case

Training Selection, Thinning of PPP

- Select a training sequence form a set $(\mathbf{t_i})$ of L_τ orthogonal training sequences randomly

$$\begin{aligned}\mathbf{t_i^H t_j} &= 0 \text{ if } i \neq j \\ &= 1 \text{ if } i = j\end{aligned}$$

for $i, j = 0$ to $L_\tau - 1$, where $\mathbf{t_i}$ is $L_\tau \times 1$ training symbol vector

- Thinning: PPP ϕ_m is divided into two independent PPPs:
  - **Interferers Using Training $\mathbf{t_o}$**: PPP $\phi_m^{t_o}$ with density $\frac{\lambda_m}{L_\tau}$
  - **Interferers Using Training other than $\mathbf{t_o}$**: PPP $\phi_m^{t_i}$ with density $\frac{\lambda_m(L_\tau - 1)}{L_\tau}$

Training Phase

- Assuming the uplink transmission to be interference limited
- WLOG assuming typical user selects training sequence $\mathbf{t_o}$
- Data $\mathbf{y}_\tau$ received at a typical BS (Slivnyak's Theorem) located at the origin 'o'

$$\mathbf{y}_\tau = \sqrt{P_\tau L_\tau l_{ou}^2} h_{ou} \mathbf{t_o} + \sum_{v \in \phi_m^{t_o} \setminus u} \sqrt{P_\tau L_\tau l_{ov}^2} h_{ov} \mathbf{t_o}$$

- $h_{ou}, h_{ov} \sim \mathcal{CN}(0,1)$
- $\mathbf{y}_\tau$ is $L_\tau \times 1$
- $l_{ou}^2 = min(1, r^{-\alpha})$ where r is the distance of the BS form the typical user u
- $l_{ov}^2 = min(1, \|v\|^{-\alpha})$ where $\|v\|$ is the distance of the interferers using training $\mathbf{t_o}$ from the BS
- α is the pathloss coefficient

Channel Estimation: LMMSE

Linear Minimum Mean Square Error Estimate (LMMSE)

- Observation Signal: $y_\tau = \mathbf{t_o^H y}_\tau$ (Scalar)
- LMMSE Estimate of h_{ou}: $\hat{h}_{ou} = \frac{\mathbb{E}[h_{ou} y_\tau^*]}{\mathbb{E}[y_\tau y_\tau^*]} y_\tau$
- We get the estimate conditioned over r (First Contact Distance) as:

$$\hat{h}_{ou} = \frac{\sqrt{P_\tau L_\tau l_{ou}^2}}{P_\tau L_\tau l_{ou}^2 + \mathbb{E}[\sum_{v \in \phi_m^{to} \setminus u} P_\tau L_\tau l_{ov}^2 |h_{ov}|^2]} y_\tau$$

Now to compute $\mathbb{E}[\sum_{v \in \phi_m^{to} \setminus u} P_\tau L_\tau l_{ov}^2 |h_{ov}|^2]$

Channel Estimate

- **Use Campbell's Theorem**

$$\mathbb{E}[\sum_{v \in \phi_m^{to} \setminus u} P_\tau L_\tau l_{ov}^2 |h_{ov}|^2] = P_\tau \pi \lambda_m \left(\frac{\alpha}{\alpha - 2} \right)$$

therefore

$$\hat{h}_{ou} = \frac{\sqrt{P_\tau L_\tau l_{ou}^2}}{P_\tau L_\tau l_{ou}^2 + P_\tau \pi \lambda_m \left(\frac{\alpha}{\alpha-2} \right)} y_\tau$$

$$\mathbb{E}[|\hat{h}_{ou}|^2] = \frac{L_\tau l_{ou}^2}{L_\tau l_{ou}^2 + \pi \lambda_m \left(\frac{\alpha}{\alpha-2} \right)}$$

Estimation Error

- Now
$$h_{ou} = \hat{h}_{ou} + \tilde{h}_{ou}$$
where $\tilde{h}_{ou}$ is the estimation error
- **Estimation Error Variance**(σ_e^2)
Using the orthogonality of $\hat{h}_{ou}$ and $\tilde{h}_{ou}$
$$\sigma_e^2 = \mathbb{E}[|\tilde{h}_{ou}|^2] = \frac{1}{1 + \frac{L_\tau l_{ou}^2}{\pi \lambda_m \left(\frac{\alpha}{\alpha-2}\right)}}$$
- Note: The pilot symbols are getting corrupted by the MUs using same training sequence as typical user 'u'.
- The considered model inherently captures the effect of **Pilot Contamination**

λ_m vs σ_e for different L_τ

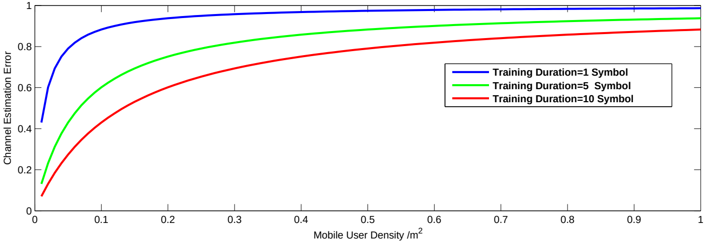

Figure : Plot of λ_m vs σ_e, for $\alpha = 3$, distance $r = 2$ and $L_\tau = 1, 5$ and 10

- Higher the MU density, higher the pilot contamination
- High estimation error

L_τ vs σ_e for different λ_m

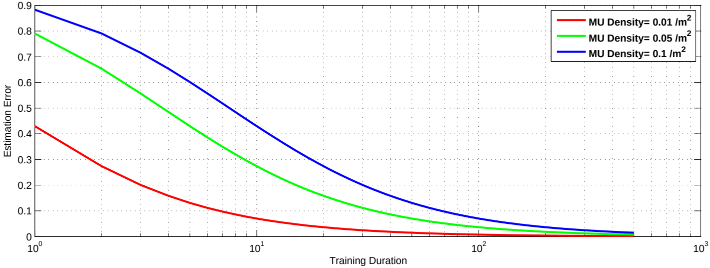

Figure : Plot of L_τ vs σ_e, for $\alpha = 3$, distance $r = 2$ and $\lambda_m = 0.01, 0.05$ and 0.1

- As L_τ increases, thinning of PPP takes place
- Hence, pilot Contamination decreases and estimate becomes more accurate

Downlink Transmission

- Suppose the typical MU to be at the origin 'o'
- BS 'b' will now transmit and set of BS 'y' will interfere
- The signal y_D received at the MU will be

$$y_D = \sqrt{P_D l_{bo}^2} h_{bo} s_{bo} + \sum_{y \in \phi_B \backslash b} \sqrt{P_D l_{yo}^2} h_{yo} s_{yo} + n$$

  - $h_{bo}, h_{yo} \sim \mathcal{CN}(0, 1)$
  - $h_{bo} = h_{ou}$ (reciprocity), also $l_{bo}^2 = l_{ou}^2$
  - $l_{yo}^2 = min(1, \|y\|^{-\alpha})$ where $\|y\|$ is the distance of the interfering Base Stations form the MU
  - Assume n to be AWGN $\sim \mathcal{CN}(0, \sigma_n^2)$

Precoding

- BS sends precoded signal as:

$$s_{bo} = \frac{\hat{h}_{ou}^*}{|\hat{h}_{ou}|} x_{bo}$$

- Using

$$h_{ou} = \hat{h}_{ou} + \tilde{h}_{ou}$$

- Therefore, the signal becomes

$$y_D = \sqrt{P_D l_{ou}^2} |\hat{h}_{ou}| x_{ou} + \sqrt{P_D l_{ou}^2} \tilde{h}_{ou} \frac{\hat{h}_{ou}^*}{|\hat{h}_{ou}|} x_{ou} + \sum_{y \in \phi_B \setminus b} \sqrt{P_D l_{yo}^2} h_{yo} s_{yo} + n$$

Signal to Interference Plus Noise Ratio

- **Signal To Noise + Interference Ratio (SINR)**

$$SINR = \frac{P_D l_{ou}^2 |\hat{h}_{ou}|^2}{P_D l_{ou}^2 |\tilde{h}_{ou}|^2 + P_D \mathcal{I}_B + \sigma_n^2}$$

where

$$\mathcal{I}_B = \sum_{y \in \phi_B \setminus b} l_{yo}^2 |h_{yo}|^2$$

- Note: The above *SINR* expression is a Random Variable
- Here, we can analyse the system in two ways
  - Coverage Probability Analysis
  - Worst Case Capacity: Using Worst Case Noise Theorem
- Here, we perform the analysis on Coverage Probability

Coverage conditioned on r

- **Probability of Coverage ($P_{c|r}$), conditioned over the nearest neighbour distance (r)**

$$
\begin{aligned}
P_{c|r} &= \mathbb{P}(SINR > \theta|r) \\
&= \mathbb{P}\left(\frac{P_D l_{ou}^2 |\hat{h}_{ou}|^2}{P_D l_{ou}^2 |\tilde{h}_{ou}|^2 + P_D \mathcal{I}_B + \sigma_n^2} > \theta \middle| r\right)
\end{aligned}
$$

$$
= \mathbb{P}\left(|\bar{h}_{ou}|^2 > \frac{\theta}{(1-\sigma_e^2)}|\tilde{h}_{ou}|^2 + \frac{\theta}{l_{ou}^2(1-\sigma_e^2)}\mathcal{I}_B + \frac{\theta\sigma_n}{P_D l_{ou}^2(1-\sigma_e^2)} \middle| r\right)
$$

- Normalizing $|\hat{h}_{ou}|^2$ as

$$
|\hat{h}_{ou}|^2 = (1-\sigma_e^2)|\bar{h}_{ou}|^2
$$

where, $\bar{h}_{ou} \sim \mathcal{CN}(0,1)$ and $|\bar{h}_{ou}|^2$ is exponentially distributed

- We get

$$P_{c|r} = \mathbb{E}\left[exp\left(-\frac{\theta}{(1-\sigma_e^2)}|\tilde{h}_{ou}|^2\right)\right]\mathbb{E}\left[exp\left(-\frac{\theta}{l_{ou}^2(1-\sigma_e^2)}\mathcal{I}_B\right)\right] exp\left(-\frac{\theta\sigma_n}{P_D l_{ou}^2(1-\sigma_e^2)}\right)$$

- Note: All the expectations are conditioned on r
- Conditioned on r, $exp\left(-\frac{\theta\sigma_n}{P_D l_{ou}^2(1-\sigma_e^2)}\right)$ is a constant
- Calculating each of the terms one by one

First Term

- **First Term** $= \mathbb{E}\left[exp\left(-\frac{\theta}{(1-\sigma_e^2)}|\tilde{h}_{ou}|^2\right)\right]$
- Laplace transform of $|\tilde{h}_{ou}|^2$ evaluated at $s = -\frac{\theta}{(1-\sigma_e^2)}$

$$\textbf{First Term} = \frac{1}{1 + \frac{\theta\sigma_e^2}{1-\sigma_e^2}}$$

- Using,

$$|\tilde{h}_{ou}|^2 = \sigma_e^2|\bar{h}'_{ou}|^2$$

where, $\bar{h}'_{ou} \sim \mathcal{CN}(0,1)$ and $|\bar{h}'_{ou}|^2$ is exponentially distributed

Second Term

- **Second Term** $= \mathbb{E}\left[exp\left(-\frac{\theta}{l_{ou}^2(1-\sigma_e^2)}\mathcal{I}_B\right)\right]$
- Replace $\mathcal{I}_B = \sum_{y\in\phi_B\backslash b} l_{yo}^2|h_{yo}|^2$, we get

$$\textbf{Second Term} = \mathbb{E}\left[\prod_{y\in\phi_B\backslash b} exp\left(-\frac{\theta}{l_{ou}^2(1-\sigma_e^2)} l_{yo}^2|h_{yo}|^2\right)\right]$$

$$\stackrel{(a)}{=} \mathbb{E}\left[\prod_{y\in\phi_B\backslash b} \mathbb{E}_{h_{yo}} exp\left(-\frac{\theta}{l_{ou}^2(1-\sigma_e^2)} l_{yo}^2|h_{yo}|^2\right)\right]$$

Second Term

$$\stackrel{(b)}{=} \mathbb{E}\left[\prod_{y\in\phi_B\setminus b} \frac{1}{1+\frac{\theta l_{yo}^2}{l_{ou}^2(1-\sigma_e^2)}}\right]$$

where, (a) follows from independence of h_{yo} and (b) follows from the laplace transform of $|h_{yo}|^2$ evaluated at $s = -\frac{\theta l_{yo}^2}{l_{ou}^2(1-\sigma_e^2)}$

- **Apply PGFL: Products over PPP**

$$\textbf{Second Term} = exp\left(-\lambda_B \int_{\mathbb{R}^2}\left(1 - \frac{1}{1+\frac{\theta l_{yo}^2}{l_{ou}^2(1-\sigma_e^2)}}\right) dy\right)$$

Final Expression

$$\textbf{Second Term} = exp\left(-2\pi\lambda_B \int_r^{\infty}\left(\frac{1}{1+\frac{l_{ou}^2(1-\sigma_e^2)}{\theta l_{yo}^2}}\right) ydy\right)$$

- Note that the integral is from r to ∞
- As there are no BS closer than the tagged BS
- **Probability of Coverage conditioned on** r

$$P_{c|r} = \textbf{First Term} \times \textbf{Second Term} \times exp\left(-\frac{\theta\sigma_n}{P_D l_{ou}^2(1-\sigma_e^2)}\right)$$

Final Expression: Unconditioning on r

- Unconditioning over r
- The PDF of r is (First Contact Distance Distribution)

$$f_R(r) = 2\pi r \lambda_B exp(-\pi r^2 \lambda_B)$$

- **Probability of Coverage (P_c)**

$$P_c = \int_0^\infty P_{c|r} f_R(r) dr$$

L_τ vs P_c

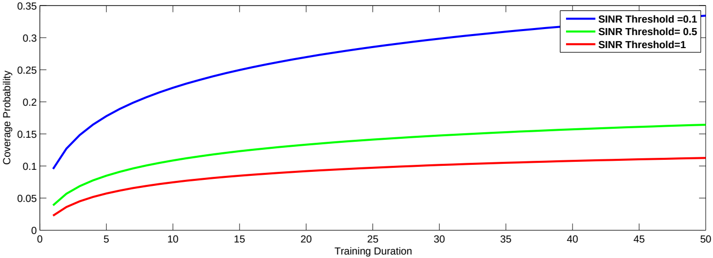

Figure : Plot of L_τ vs P_c, for $\alpha = 3$, $\lambda_m = 0.1$, $\lambda_B = 0.01$, for *SINR* Threshold $\theta = 0.1, 0.5$ and 1, $\frac{P_D}{\sigma_n} = 20$

- The coverage probability increases with L_τ
- Interference making the coverage to saturate for large training durations

$SINR$ Threshold θ vs P_c

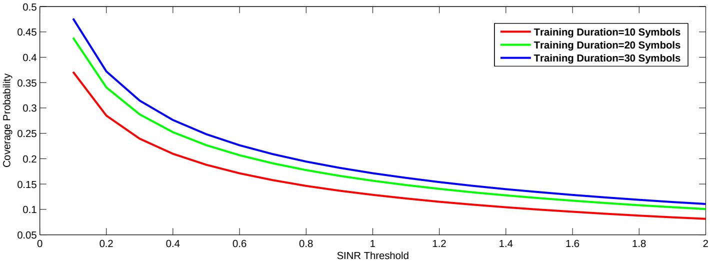

Figure : Plot of $SINR$ threshold θ vs P_c, for $\alpha = 3$, $\lambda_m = 0.1$, $\lambda_B = 0.02$, $L_\tau = 10, 20$ and 30 and $\frac{P_D}{\sigma_n} = 20$

SNR vs P_c

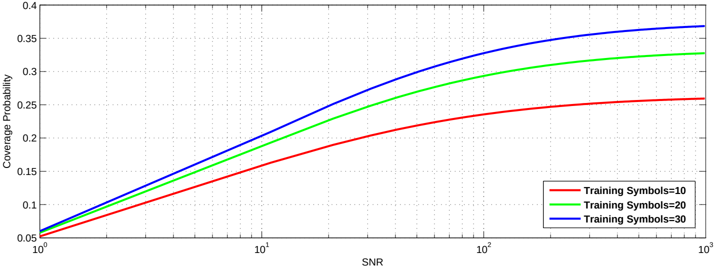

Figure : Plot of SIR vs P_c, for $\alpha = 3$, $\lambda_m = 0.1$, $\lambda_B = 0.02$, $L_\tau = 10, 20$ and 30, $\theta = 0.5$

- Only one of the three terms depends on P_D
- P_c saturates with increasing P_D

SNR vs P_c

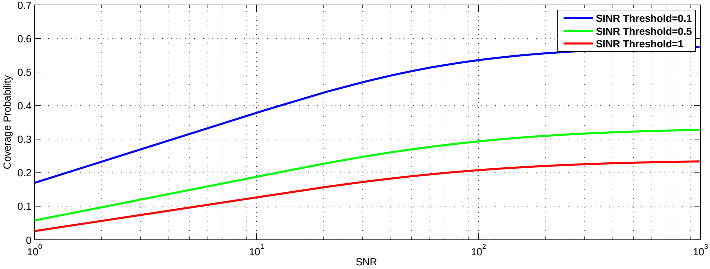

Figure : Plot of SIR vs P_c, for $\alpha = 3$, $\lambda_m = 0.1$, $\lambda_B = 0.02$, $L_\tau = 20$, $\theta = 0.1, 0.5$ and 1

- Again, P_c saturates with increasing P_D

Comparison with Perfect Channel Estimate

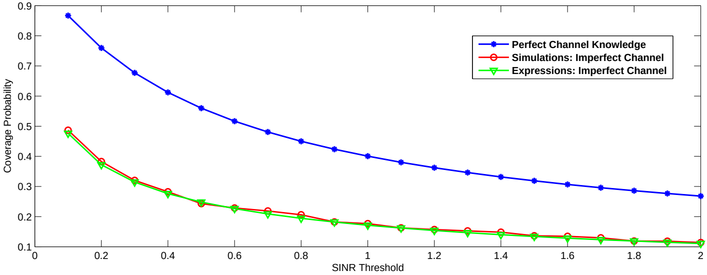

Figure : Comparison of P_c with $SINR$ threshold θ for perfect and imperfect channel knowledge, for $\alpha = 3$, $\lambda_m = 0.1$, $\lambda_B = 0.02$, $L_\tau = 30$ and $\frac{P_D}{\sigma_n} = 20$

Future Extensions

- Multi-tier heterogeneous networks
- Study for cell edge users
- Uplink Channel Study
- Connectivity based on SINR
- Dependency between ϕ_B and ϕ_m

Thank You