Error Exponent Analysis for Bayesian Energy Detection under Fading

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Outline

- Motivation and System Model.
- *Error Exponents with a confidence level.*
- Detection at the Sensor level.
- Detection at the Fusion Center.
  - Lower bounds on the error exponents.
  - Actual error exponents.
- Application : Narrowband vs. Wideband Sensing.
- Numerical and Simulation results.

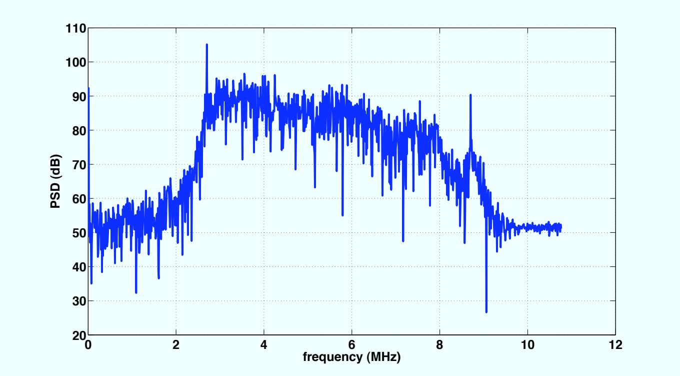
System Model

- N sensors with M observations each.
- At the individual sensors under low SNR, $V_y \triangleq \frac{1}{M}\sum_{i=1}^{M}|y_i|^2 - 1$, and

$$\mathcal{H}_0 \quad : \quad V_y \sim \mathcal{N}\left(0, \frac{1}{M}\right),$$
$$\mathcal{H}_1 \quad : \quad V_y \sim \mathcal{N}\left(|h|^2 P, \frac{1}{M}\right).$$

- Interest : Wideband (WB) primary signals with a strong pilot.
- NB signals undergo Rayleigh fading, WB signals undergo Lognormal fading [Shellhammer et al. 2006]

Primary Signals of Interest - An Example



The Bayesian ED Problem

- Bayesian problem $\Rightarrow$ minimize p_e. LRT is optimal.
- $LR(V_y) = \dfrac{\frac{1}{\sqrt{2\pi/M}} \int_{|h|^2} \exp\left(-\frac{M(V_y - |h|^2 P)^2}{2}\right) f_{|h|^2}(|h|^2) d|h|^2}{\frac{1}{\sqrt{2\pi/M}} \exp\left(-\frac{MV_y^2}{2}\right)}$.
- Closed form analysis of p_e becomes difficult.

Error Exponents

- Error exponent gives the exponential rate of decay on p_e. Mathematically,

$$\epsilon_e \triangleq -\lim_{M\to\infty} \frac{\log p_e}{M} \quad \text{and} \quad \epsilon_E^{(N)} \triangleq -\lim_{M\to\infty} \frac{\log P_E}{M}.$$

- Turns out that the error exponents under both WB and NB sensing are zero. Therefore, in terms of the error exponents, WB and NB sensing problems are equivalent.
- Discount the low channel gains?

Error Exponent With A Confidence Level (EECL)

Definition

Let S_q denote a set of channel instantiations such that $P(h \in S_q) = q$. The highest error exponent achievable over all possible choices of S_q is the error exponent with confidence q.

- This novel concept is used to answer the question of WB vs. NB SS.

EECL at the Sensor Level

Theorem

For the above SS problem, a positive error exponent of $(|h_0|^2 P)^2/8$ is achievable with a confidence level q, where $|h_0|^2$ satisfies $\mathcal{P}\left(|h|^2 > |h_0|^2\right) = q$.

- Note that the above is valid for a general fading model. Also, note that $EECL = 0$, with $q = 1$.
- It follows that,
  - For Rayleigh fading case, an EECL of $\frac{(\alpha_0 P)^2}{8}$ is achievable with confidence $\exp(-\alpha_0)$.
  - For lognormal shadowing case, an EECL of $\frac{(\ell_0 P)^2}{8}$ is achievable with confidence $1 - Q\left(\frac{\log(\ell_0/P)}{\sigma_s}\right)$.

Outline of the Proof

- Let $\alpha = |h|^2$. It is straightforward to show that

$$p_e = \pi_0 Q(x\sqrt{M}) + \pi_1 \int_{-\infty}^{x} \int_{\alpha_0}^{\infty} f_{\mathcal{N}}\left(v - \alpha P, \frac{1}{\sqrt{M}}\right) \frac{f_\alpha(\alpha)}{q} d\alpha dv.$$

- Further simplification yields

$$\frac{q\pi_0}{\pi_1} = \int_{\alpha_0}^{\infty} \exp\left(M\left(x\alpha P - \frac{\alpha^2 P^2}{2}\right)\right) f_\alpha(\alpha) d\alpha.$$

- The rest of the proof involves showing that $x \to \frac{\alpha_0 P}{2}$ as $M \to \infty$ and examining the exponent of p_f (obtained through direct analysis) which is $\frac{x^2}{2}$.

- Upper bound : Let $g(x,\alpha) \triangleq x\alpha P - \frac{\alpha^2 P^2}{2}$. Observe that $g\left(\frac{\alpha_0 P}{2}, \alpha\right) < 0$, for $\alpha \geq \alpha_0$. For $x_M < \frac{\alpha_0 P}{2}$, $g(x_M,\alpha) < 0$, for $\alpha \geq \alpha_0$. Let $g_{\max} \triangleq \max_{\alpha \geq \alpha_0} g(x_M,\alpha)$. $\Rightarrow$ The integral would then $\rightarrow$ zero.
- Lower bound : Let $x_0 > \frac{\alpha_0 P}{2}$. Therefore, $g(x_0,\alpha) > 0$, when $\alpha < \frac{2x_0}{P}$. By assumption, $\alpha_0 \leq \alpha$. $\Rightarrow$, $g(x_0,\alpha) > 0$, when $\alpha_0 \leq \alpha < \frac{2x_0}{P}$. Therefore, for $x_M > x_0$, $g(x_M,\alpha) > 0$. There exists a $\delta > 0$, such that $g(x_M,\alpha) > 0$, for $\alpha_0 \leq \alpha < \frac{2x_0}{P} - \delta$. Let $g_{\min} \triangleq \min_{\alpha_0 \leq \alpha < \frac{2x_0}{P} - \delta} g(x_M,\alpha)$. $\Rightarrow$ The integral would then $\rightarrow +\infty$.

EECL at the FC

- Question : Can EECL be improved with a decentralized detection scheme?
- OR rule is considered for analytical tractability. The FC should exploit even if one of the N sensors experiences a "good" channel state.

The analysis

- Obtain exponent on P_F (direct) and the same on P_M (multinomial expansion and analyzing term-by-term) i.e.,
- $\epsilon_F = \alpha_{\min}^2 P/8$ and $\epsilon_M = \sum_{j=1}^{N} \frac{1}{2}\left(\alpha_j P - \frac{\alpha_{\min} P}{2}\right)^2 \mathbb{I}_{\left\{\frac{2\alpha_j}{\alpha_{\min}} > 1\right\}}$.
- Equating both and bringing in the notion of confidence, $\mathcal{P}\left\{\sum_{j=1}^{N}\left(\alpha_j - \frac{\alpha_{\min}}{2}\right)^2 \mathbb{I}_{\left\{\frac{2\alpha_j}{\alpha_{\min}} > 1\right\}} \le \frac{\alpha_{\min}^2}{4}\right\} = 1 - q$.

LB on EECL using OR rule for Rayleigh fading

Theorem

When the channel between the primary and sensors is Rayleigh faded, given that the FC combines decisions from N sensors using the OR rule, the error exponent with confidence level q at the FC for the HT is lower bounded by $(\alpha_{min}P)^2/8$ *with confidence q, where* α_{min} *satisfies*

$$\alpha_{min} = 2\left(\frac{1-q}{C_N}\right)^{\frac{1}{N}}, \quad C_N \triangleq \sum_{k=0}^{N} \binom{N}{k} \frac{\mathcal{V}_k}{2^k},$$

where $\mathcal{V}_k = \pi^{k/2}/\Gamma\left(1+\frac{k}{2}\right)$ *is the volume of a k dimensional unit sphere.*

LB on EECL using OR rule for Lognormal fading

Theorem

When the channel between the primary and sensors is lognormal faded, given that the FC combines the decisions from N sensors using the OR rule, the error exponent with confidence level q at the FC for HT is lower bounded by $\frac{(\ell_{min}P)^2}{8}$ *with confidence q, where* ℓ_{min} *satisfies*

$$\sum_{k=0}^{N} \binom{N}{k} D_A^k D_B^{N-k} \frac{\mathcal{V}_k}{2^k} = 1 - q, \ \text{with } \mathcal{V}_k = \pi^{k/2} / \Gamma\left(1 + \frac{k}{2}\right)$$

$$D_A \triangleq \frac{1}{2\sigma_s\sqrt{2\pi}} \exp\left(-\frac{\left(\log\left(\frac{\ell_{min}}{P}\right)\right)^2}{2\sigma_s^2}\right), D_B \triangleq Q\left(\frac{1}{\sigma_s}\log\left(\frac{2P}{\ell_{min}}\right)\right).$$

EECL at the FC - OR rule and Rayleigh

Theorem

When the channel between the primary transmitter and sensors is Rayleigh faded, given that the FC combines decisions from N sensors using the OR rule, the error exponent on the probability of error at the FC for the above SS problem is positive and is given by $\frac{(\alpha_{min}P)^2}{8}$ *with confidence q, where* α_{min} *satisfies*

$$\sum_{l=1}^{N} \binom{N}{l} \left[1 - \exp\left(-\frac{\alpha_{min}}{2}\right)\right]^{N-l} \mathcal{P}\left\{\sum_{k=1}^{l} a_k^2 \leq 1\right\} e^{-\frac{\alpha_{min}}{2} l}$$

$$+ \left[1 - \exp\left(-\frac{\alpha_{min}}{2}\right)\right]^{N} = 1 - q,$$

where, $a_k \sim \exp\left(\frac{\alpha_{min}}{2}\right)$.

EECL at the FC - OR rule and Lognormal

Theorem

The same detector under lognormal fading achieves an error exponent of $\frac{(\ell_{min}P)^2}{8}$ *with confidence* q*, where* ℓ_{min} *satisfies*

$$\sum_{l=1}^{N}\binom{N}{l}\left[Q\left(-\frac{\mu_s+\log\left(\frac{2}{\ell_{min}}\right)}{\sigma_s}\right)\right]^{N-l}\mathcal{P}\left\{\sum_{k=1}^{l}(t_k-1)^2\leq 1\right\}$$

$$+\left[Q\left(-\frac{\mu_s+\log\left(\frac{2}{\ell_{min}}\right)}{\sigma_s}\right)\right]^{N}=1-q,$$

where, $t_k \sim \mathcal{LN}\left(\mu_s+\log\left(\frac{2}{\ell_{min}}\right),\sigma_s\right)$.

Application : WB and NB SS

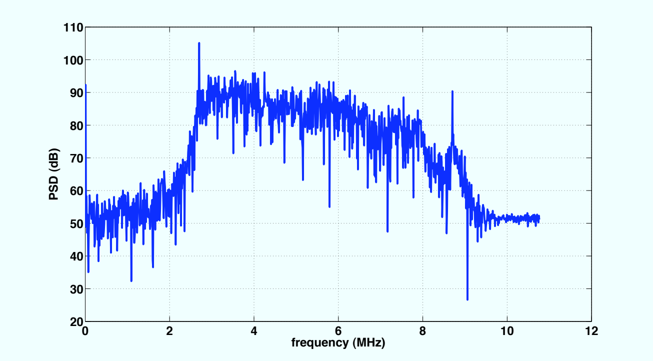


Figure: DTV Power Spectral Density

Quantifying in terms of error exponents with confidence

- Let ϵ_{NB} and ϵ_{WB} represent the error exponents by detectors under NB and WB SS, respectively.
- At the individual sensors, with the same confidence level q, NB sensing is "better" than WB sensing whenever,
  $\frac{(\alpha_0 P_{NB})^2}{8} > \frac{(l_0 P_{WB})^2}{8} \Rightarrow \left(\frac{P_{NB}}{P_{WB}}\right)^2 > \left(\frac{l_0}{\alpha_0}\right)^2.$
- Similarly, at the FC, NB sensing is better than WB sensing with the same confidence level whenever
  $\frac{(\alpha_{\min} P_{NB})^2}{8} > \frac{(\ell_{\min} P_{WB})^2}{8} \Rightarrow \left(\frac{P_{NB}}{P_{WB}}\right)^2 > \left(\frac{\ell_{\min}}{\alpha_{\min}}\right)^2.$

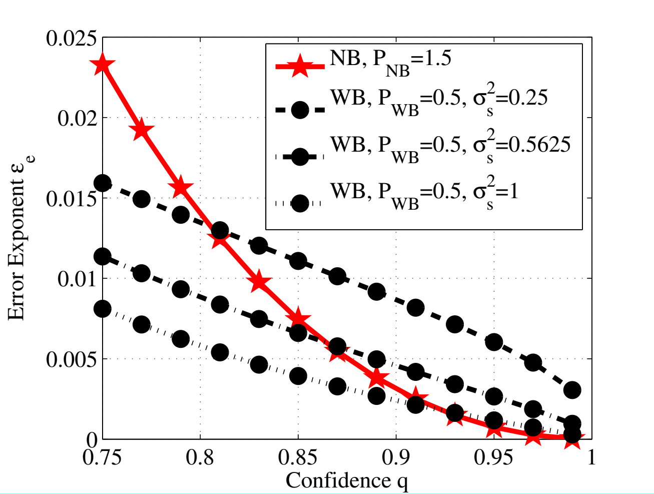


Figure: Error Exponents ϵ_e with confidence q at single sensor

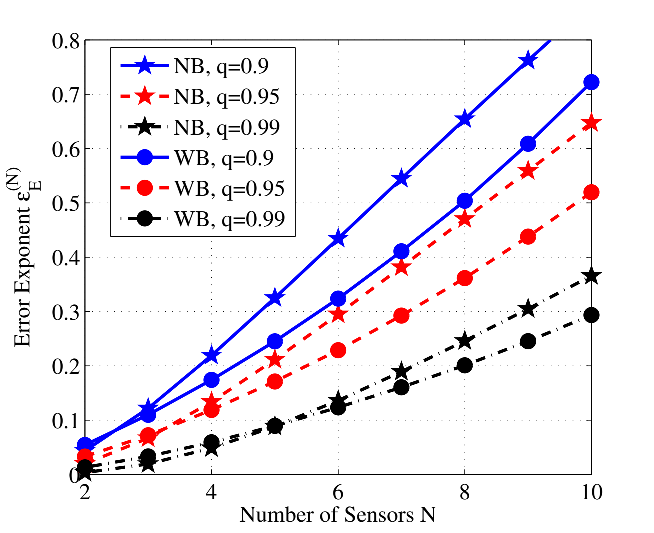


Figure: Variation of $\epsilon_E^{(N)}$ with q

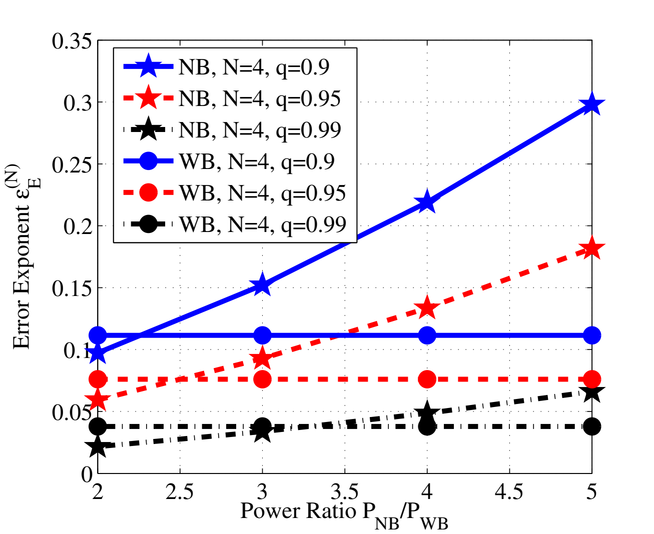


Figure: Variation of $\epsilon_E^{(N)}$ with ratio $\frac{P_{NB}}{P_{WB}}$

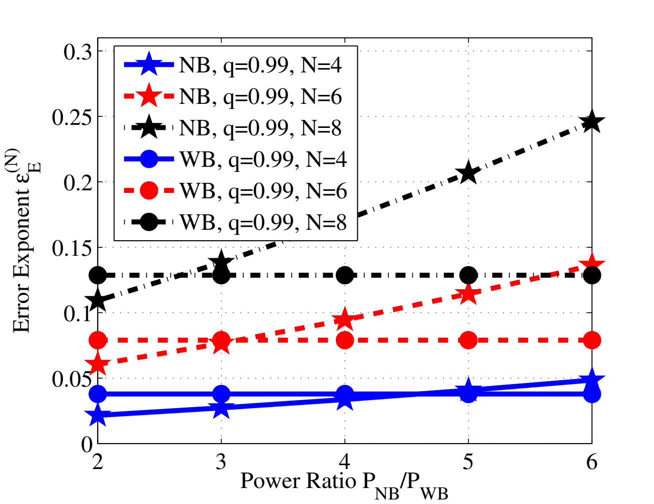


Figure: Variation of $\epsilon_E^{(N)}$ with ratio $\frac{P_{NB}}{P_{WB}}$

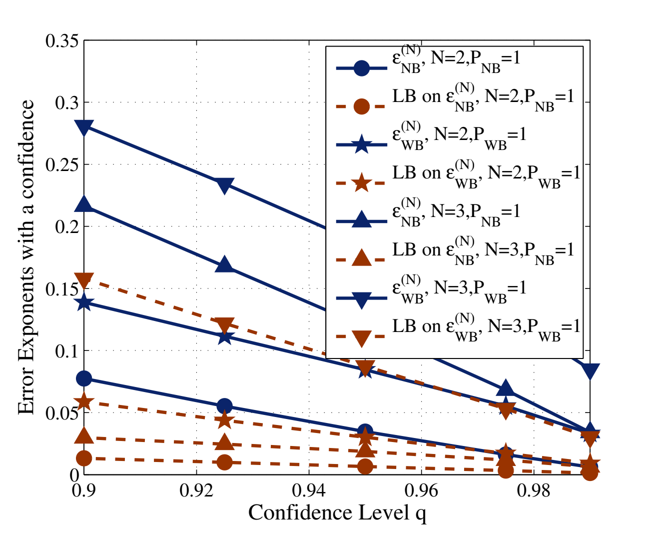


Figure: Comparision of $\varepsilon_E^{(N)}$ and its LB

List of $\alpha_{\min}$ and $\ell_{\min}$ for Different q and N

N	q=0.9			N	q=0.95			N	q=0.99	
	$\alpha_{\min}$	$\ell_{\min}$			$\alpha_{\min}$	$\ell_{\min}$			$\alpha_{\min}$	$\ell_{\min}$
2	0.3937	0.5268		2	0.2635	0.4111		2	0.1109	0.2578
3	0.6579	0.7497		3	0.4878	0.6078		3	0.2612	0.4119
4	0.8824	0.9442		4	0.6890	0.7799		4	0.4153	0.5502
5	1.07489	1.1200		5	0.8665	0.9358		5	0.5610	0.6771
6	1.2428	1.2875		6	1.02403	1.0819		6	0.6958	0.7949
7	1.3916	1.4500		7	1.1652	1.2235		7	0.8201	0.9062
8	1.5251	1.6062		8	1.2930	1.3602		8	0.9349	1.0144
9	1.6462	1.7655		9	1.4097	1.4971		9	1.0412	1.1209
10	1.7571	1.9230		10	1.5171	1.6308		10	1.1403	1.2257

Theoretical and Simulated Error Exponents (1/2)

P (dB)	q=0.9			P (dB)	q=0.95			P (dB)	q=0.99	
	Th	Sims			Th	Sims			Th	Sims
−3	34.69	35.912		0	32.888	34.189		5	11.364	14.495
−4	22.02	23.556		−0.5	26.6639	28.232		4	7.8913	9.3611
−5	12.488	13.633		−1	21.048	22.301		3	5.0505	7.3419
−7	5.5504	6.5754		−1.5	16.115	17.642		2	2.8409	4.3919
−10	1.3876	2.1998		−2.25	11.84	14.282		0	1.2626	2.3234

Table: ϵ_e for different q. All EECL values have to be multiplied by $\times 10^{-5}$.

Theoretical and Simulated Error Exponents (2/2)

N	q=0.9, P=-10 dB			N	q=0.95, P=-10 dB	
	Th	Sims			Th	Sims
2	1.9373	2.1102		2	0.86768	0.88057
3	5.4111	6.1154		3	2.9749	3.5199
4	9.7329	10.2832		4	5.9349	6.3426

Table: $\epsilon_E^{(N)}$ for different q. All EECL values have to be multiplied by $\times 10^{-4}$.

Conclusions

- Proposed a novel concept called *error exponent with a confidence level*. Using this, it was shown that the ED achieves a zero error exponent under a general fading model.
- This generalized concept was extended to the decentralized setup and performance under the OR rule was studied in detail.
- The question of WB vs. NB sensing was successfully answered using this novel metric.

Reference

- Sanjeev G., Chandra R. Murthy and Vinod Sharma, Error Exponent Analysis of Energy-Based Bayesian Spectrum Sensing Under Fading Channels, Journal draft.