Error Exponent Analysis for Bayesian Energy Detection under Fading

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Outline

- Motivation and System Model.
- *Error Exponents with a confidence level.*
- Detection at the Sensor level.
- Detection at the Fusion Center.
- Application : Narrowband vs. Wideband Sensing.

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System Model

- N sensors with M observations each.
- At the individual sensors, $V_y \triangleq \frac{1}{M}\sum_{i=1}^{M} |y_i|^2 - 1$ and

$$
\begin{aligned}
\mathcal{H}_0 &: \quad V_y \sim \mathcal{N}\left(0, \frac{1}{M}\right), \\
\mathcal{H}_1 &: \quad V_y \sim \mathcal{N}\left(|h|^2 P, \frac{1}{M}\right).
\end{aligned}
$$

- Fading can be modelled by various random variables depending on various conditions. The simplest models are Rayleigh (NB) and Lognormal (WB).

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Definition

Let S_q denote a set of channel instantiations such that $P(|h|^2 \in S_q) = q$. The highest error exponent achievable over all possible choices of S_q is defined to be the error exponent with a confidence q.

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Theorem

Let $\alpha \triangleq |h|^2$. The hypothesis test (defined earlier) achieves a positive error exponent of $(\alpha_0 P)^2/8$ with a confidence level q, where α_0 satisfies $Pr(\alpha > \alpha_0) = q$.

- **Proof.**

$$\pi_0 Q(x\sqrt{M}) + \pi_1 \int_{-\infty}^{x} \int_{\alpha_0}^{\infty} f_{\mathcal{N}}\left(v - \alpha P, \frac{1}{\sqrt{M}}\right) \frac{f_\alpha(\alpha)}{q} d\alpha dv,$$

Differentiating above equation w.r.t. x and equating to zero gives

$$\frac{q\pi_0}{\pi_1} = \int_{\alpha_0}^{\infty} \exp\left(M\left(x\alpha P - \frac{\alpha^2 P^2}{2}\right)\right) f_\alpha(\alpha) d\alpha$$

- Upper bound : Let $g(x,\alpha) \triangleq x\alpha P - \frac{\alpha^2 P^2}{2}$. Observe that $g\left(\frac{\alpha_0 P}{2}, \alpha\right) < 0$, for $\alpha \geq \alpha_0$. For $x_M < \frac{\alpha_0 P}{2}$, $g(x_M, \alpha) < 0$, for $\alpha \geq \alpha_0$. Let $g_{\max} \triangleq \max_{\alpha \geq \alpha_0} g(x_M, \alpha)$. $\Rightarrow$ The integral would then $\rightarrow$ zero.
- Lower bound : Let $x_0 > \frac{\alpha_0 P}{2}$. Therefore, $g(x_0, \alpha) > 0$, when $\alpha < \frac{2x_0}{P}$. By assumption, $\alpha_0 \leq \alpha$. $\Rightarrow$, $g(x_0, \alpha) > 0$, when $\alpha_0 \leq \alpha < \frac{2x_0}{P}$. Therefore, for $x_M > x_0$, $g(x_M, \alpha) > 0$. There exists a $\delta > 0$, such that $g(x_M, \alpha) > 0$, for $\alpha_0 \leq \alpha < \frac{2x_0}{P} - \delta$. Let $g_{\min} \triangleq \min_{\alpha_0 \leq \alpha < \frac{2x_0}{P} - \delta} g(x_M, \alpha)$. $\Rightarrow$ The integral would then $\rightarrow +\infty$.

Space for Notes

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- Motivation and System Model.
- *Error Exponents with a confidence level.*
- Detection at the Sensor level. $\epsilon_e = \frac{(|h|_0^2 P)^2}{8}$ w.c. $q = P\{|h|^2 > |h|_0^2\}$
- Detection at the Fusion Center.
  - K out of N rule and its "weakness".
  - "Remedy" : OR rule as a special case.
- Application : Narrowband vs. Wideband Sensing.

The K out of N rule

- $$K_{opt} = \min\left(N, \left\lceil \frac{\log\left(\frac{\pi_0}{1-\pi_0}\right) + N\log\left(\frac{1-p_f}{p_m}\right)}{\log\left\{\left(\frac{1-p_m}{p_f}\right)\left(\frac{1-p_f}{p_m}\right)\right\}} \right\rceil\right)$$
- It can be shown that, $\lim_{M\to\infty} K_{opt} = \frac{\epsilon_m}{\epsilon_f+\epsilon_m}N$, $\lim_{M\to\infty}(N - K_{opt}) = \frac{\epsilon_f}{\epsilon_f+\epsilon_m}N$
- Also, it can be shown $\lim_{M\to\infty}\frac{\log P_E}{M} \triangleq \epsilon_E^{(N)} = \frac{\epsilon_f\epsilon_m}{\epsilon_f+\epsilon_m}N = \frac{N\epsilon_f}{2} = \frac{N\epsilon_m}{2}$
- Rayleigh fading case, $q = e^{-\alpha_0}$. For a same confidence level, straightforward to show that $\epsilon_E^{(N)} = (P\log q)^2/(16N)$. Decentralized detector with $N \geq 2$ performs poorer in the error exponent sense!

Remedy: OR rule for Rayleigh fading

Theorem

When the channel between the primary and sensors is Rayleigh distributed, given that the FC combines decisions from N sensors using the OR rule, the error exponent with confidence level q at the FC for the HT is lower bounded by $(\alpha_{min}P)^2/8$ with confidence q, where α_{min} satisfies

$$\alpha_{min} = 2\left(\frac{1-q}{C_N}\right)^{\frac{1}{N}}, \quad C_N \triangleq \sum_{k=0}^{N} \binom{N}{k} \frac{\mathcal{V}_k}{2^k},$$

where $\mathcal{V}_k = \pi^{k/2}/\Gamma\left(1+\frac{k}{2}\right)$ is the volume of a k dimensional unit sphere.

Space for Notes

Remedy: OR rule for Lognormal Shadowing

Theorem

When the channel between the primary and sensors is lognormal faded, given that the FC combines the decisions from N sensors using the OR rule, the error exponent with confidence level q at the FC for HT is lower bounded by $\frac{(\ell_{min}P)^2}{8}$ *with confidence q, where* ℓ_{min} *satisfies*

$$\sum_{k=0}^{N} \binom{N}{k} D_A^k D_B^{N-k} \frac{\mathcal{V}_k}{2^k} = 1 - q, \ \ \textit{with}\ \mathcal{V}_k = \pi^{k/2} / \Gamma\left(1 + \frac{k}{2}\right)$$

$$D_A \triangleq \frac{1}{2\sigma_s\sqrt{2\pi}} \exp\left(- \frac{\left(\log\left(\frac{\ell_{min}}{P}\right)\right)^2}{2\sigma_s^2} \right), D_B \triangleq Q\left(\frac{1}{\sigma_s} \log\left(\frac{2P}{\ell_{min}}\right)\right).$$

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- Detection at the Fusion Center. $\epsilon_E^{(N)} = \frac{(\alpha_{\min} P)^2}{8}$ w.c. $q = P\{|h|^2 > |h|_0^2\}$
- Application : Narrowband vs. Wideband Sensing.

Application : WB and NB SS

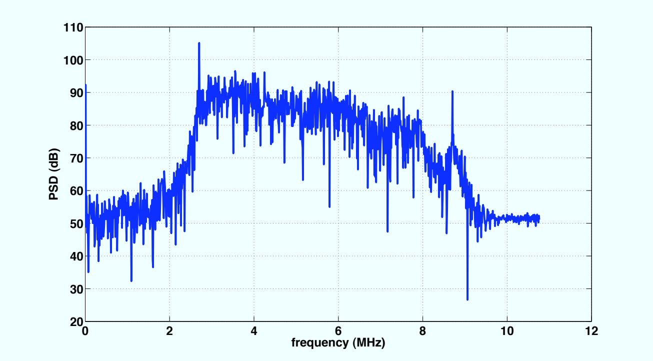

Figure: DTV Power Spectral Density

Quantifying in terms of error exponents with confidence

- Let ϵ_{NB} and ϵ_{WB} represent the error exponents by detectors under NB and WB SS, respectively.
- At the individual sensors, with the same confidence level q, NB sensing is "better" than WB sensing whenever,
$$\frac{(\alpha_0 P_{NB})^2}{8} > \frac{(l_0 P_{WB})^2}{8} \Rightarrow \left(\frac{P_{NB}}{P_{WB}}\right)^2 > \left(\frac{l_0}{\alpha_0}\right)^2.$$
- Similarly, at the FC, NB sensing is better than WB sensing with the same confidence level whenever
$$\frac{(\alpha_{\min} P_{NB})^2}{8} > \frac{(\ell_{\min} P_{WB})^2}{8} \Rightarrow \left(\frac{P_{NB}}{P_{WB}}\right)^2 > \left(\frac{\ell_{\min}}{\alpha_{\min}}\right)^2$$

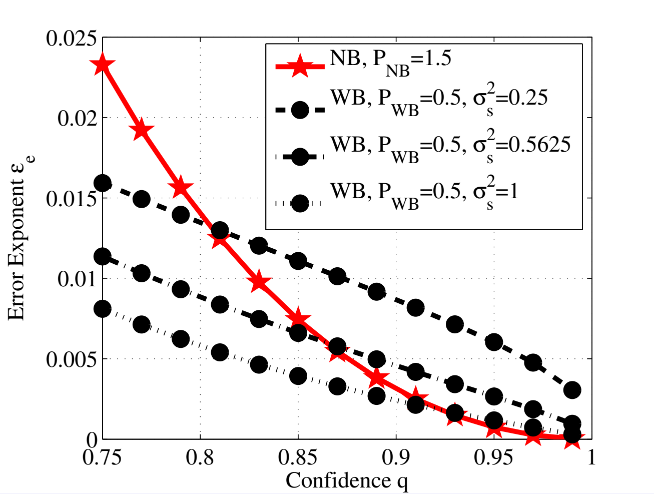

Figure: Error Exponents ϵ_e with confidence q at single sensor

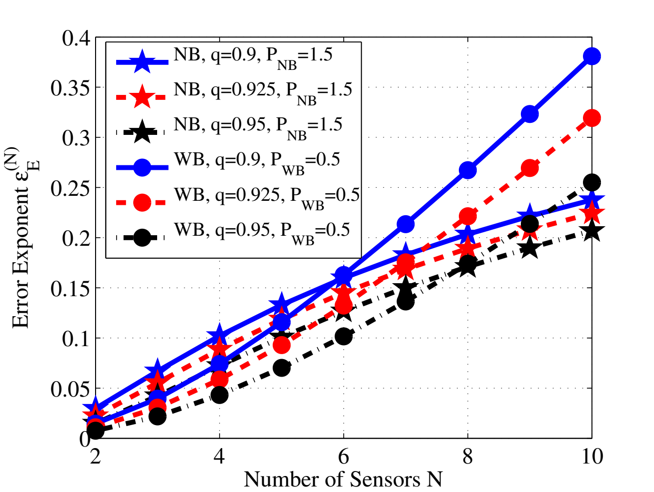

Figure: Variation of $\epsilon_E^{(N)}$ with q

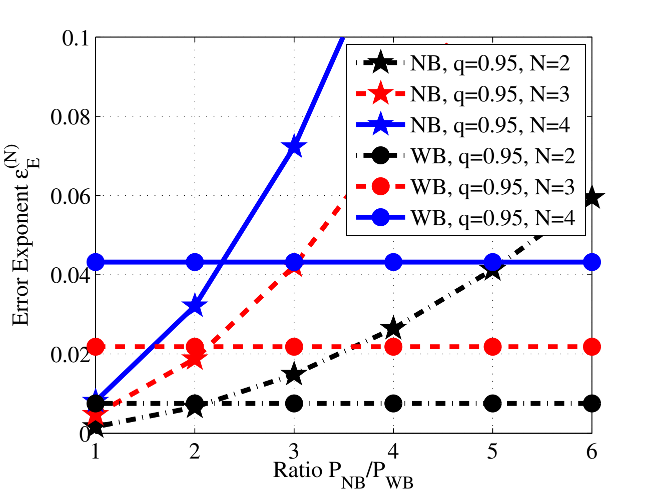

Figure: Variation of $\epsilon_E^{(N)}$ with ratio $\frac{P_{NB}}{P_{WB}}$

Conclusions

- When the channel between primary transmitter and the sensors is faded, the error exponent on the probability of error is zero (for most of the practically used channel models).
- A novel concept viz. the error exponent with a confidence level was introduced and used to compare different detection schemes.
- Error exponents with a confidence level at the FC for the K out of N rule and the OR rule were derived.

Reference

- Sanjeev G., Chandra R. Murthy and Vinod Sharma, Error Exponent Analysis of Energy-Based Bayesian Spectrum Sensing Under Fading Channels, submitted to globecom 2011