Frame Theory

Part I: Introduction

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Outline

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Finite Dimensional Vector Space

- Any element in a finite dimensional vector space has representation in terms of its basis vectors.

$$\mathbf{x} = \sum_{i=1}^{N} a_i \mathbf{e}_i$$

where $\mathbf{e}_i$ are the N basis vectors.

  - The N basis vectors may be orthogonal, but they must be independent for unique representation.
  - Orthonormal basis vectors ensure norm is preserved as well as representation is unique.
  - e.g. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$.
  - e.g. For $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{e}_1 = \{1, 0\}$, $\mathbf{e}_2 = \frac{1}{\sqrt{2}}\{1, 1\}$

Analysis and Synthesis

- The decomposition of $\mathbf{x}$ in terms of components of $\mathbf{e}_i$'s is *Analysis*. i.e.,

$$\mathbf{a} = \mathbf{T}\mathbf{x}$$

  - $\mathbf{T} = [\mathbf{e}_1^T \mathbf{e}_2^T \ldots \mathbf{e}_N^T]$ denotes the linear transformation from $\mathbb{R}^N$ to $\mathbb{R}^N$.

- The re-computation of $\mathbf{x}$ from the coefficients of the representation is *Synthesis*.

$$\mathbf{x} = \mathbf{T}^{-1}\mathbf{a}$$

Bi-orthonormal Basis

- If a given vector $\mathbf{x}$ can be represented as

$$\mathbf{x} =< \mathbf{x}, \mathbf{e}_1 > \tilde{\mathbf{e}}_1 + < \mathbf{x}, \mathbf{e}_2 > \tilde{\mathbf{e}}_2$$

  then $\{\mathbf{e}_1, \mathbf{e}_2\}, \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ are called Bi-orthonormal basis vectors.

  - The vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ are the row vectors of $\mathbf{T}$, *Analysis* matrix.
  - The vectors $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2\}$ are the column vectors of $\mathbf{T}^{-1}$, *Synthesis* matrix.

- In the case of orthogonal Analysis matrix, $\mathbf{T}^{-1} = \mathbf{T}^H$.

Overcomplete Representations

- If the basis vectors are not independent, the representation is not unique.
  - Number of vectors in the basis is more than the dimension of the vector space.
  - This basis is said to be overcomplete.
- The norm in the representation need not match the norm of the original vector.
  - e.g. Consider repeated basis vectors $\{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \ldots, \mathbf{e}_N, \mathbf{e}_N\}$. The norm in the representation is twice that of the original vector $\mathbf{x}$.

Overcomplete Representations : An example

- Consider the following basis vectors for $\mathbb{R}^2$,
  $\mathbf{g}_1 = [10]^T, \mathbf{g}_2 = [01]^T, \mathbf{g}_3 = [1-1]^T$.
  - $\mathbf{c} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}$
  - Since $\mathbf{T}$ has many inverses, there is more than one set of bi-orthogonal basis vectors.
  - One of them is

$$\mathbf{x} = <\mathbf{x}, \mathbf{g}_1> 2\mathbf{g}_1 + <\mathbf{x}, \mathbf{g}_2> (\mathbf{g}_2 - \mathbf{g}_1) - <\mathbf{x}, \mathbf{g}_3> \mathbf{g}_1$$

- This redundant set of vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is called a frame.
- The set $\{\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_3\}$ is called a dual frame.

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Formal Definition

- A set of elements $\{\mathbf{g}_k\}, k \in \mathcal{K}, \mathbf{g}_k \in \mathcal{H}$ is called a frame for the Hilbert space $\mathcal{H}$ if

$$A\|\mathbf{x}\|_2^2 \leq \sum_{k \in \mathcal{K}} | < \mathbf{x}, \mathbf{g}_k > |^2 \leq B\|\mathbf{x}\|^2$$

where $A, B \in \mathbb{R}$ and $0 < A \leq B < \infty$.

  - The constants A, B are called frame bounds
- If $A = B$, then the frame is called a tight frame.
  - Fourier transform is a tight frame

Examples

- Let $\{\mathbf{e}_k\}, k = 1, 2, \ldots, \infty$ be the orthonormal basis for infinite dimensional Hilbert space $\mathcal{H}$.
  - By repeating each element, we get a frame with frame bounds $A = B = 2$.
- Consider another frame
  $\{\mathbf{g}_k\}_{k=1}^{\infty} = \left\{\mathbf{e}_1, \frac{1}{\sqrt{2}}\mathbf{e}_2, \frac{1}{\sqrt{2}}\mathbf{e}_2, \frac{1}{\sqrt{3}}\mathbf{e}_3, \frac{1}{\sqrt{3}}\mathbf{e}_3, \frac{1}{\sqrt{3}}\mathbf{e}_3, \ldots\right\}.$
  - This is a tight frame

Frame Bounds

- The condition for a frame can be written as

$$A\|\mathbf{x}\|^2 \leq \|\mathbf{T}\mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2$$

$$\lambda_{min}(\mathbf{T}^T\mathbf{T})\|\mathbf{x}\|^2 \leq \mathbf{x}^H\mathbf{T}^H\mathbf{T}\mathbf{x} \leq \lambda_{min}(\mathbf{T}^T\mathbf{T})\|\mathbf{x}\|^2$$

  where $\mathbf{T}$ represents the Analysis operator, $\mathbf{T} : \mathcal{H} \rightarrow \mathbb{R}^d$.
- By definition, $\mathbf{T}$ is linear and left-invertible
- Existence of lower frame bound A guarantees $\mathbf{T}$ is left-invertible and $\{\mathbf{g}_k\}$ is complete.
- Existence of upper frame bound B guarantees $\mathbf{T}\mathbf{x}$ is bounded.

Canonical Dual Frame

- For a finite dimensional vector $\mathbf{x}$, $\mathbf{c} = \mathbf{T}\mathbf{x}$.
    - That is, $\mathbf{x} = \mathbf{T}^{\dagger}\mathbf{c}$
    - $\mathbf{x} = \left(\mathbf{T}^H\mathbf{T}\right)^{-1}\mathbf{T}^H\mathbf{c}$
    - $\mathbf{x} = \sum_k \tilde{\mathbf{g}}_k c_k$
    - $\Rightarrow \tilde{\mathbf{g}}_k = \left(\mathbf{T}^H\mathbf{T}\right)^{-1}\mathbf{g}_k$
- This set $\{\tilde{\mathbf{g}}_k\}$ is called canonical dual of $\{\mathbf{g}_k\}$.

Frame operator

- Let $\{\mathbf{g}_k\}_{k\in\mathcal{K}}$ be a frame for the Hilbert space $\mathcal{H}$.
- The operator $\mathbb{S}:\mathcal{H}\rightarrow\mathcal{H}$ defined as $\mathbb{S}=\mathbf{T}^H\mathbf{T}$

$$\mathbb{S}\mathbf{x}=\sum_{k\in\mathcal{K}} <\mathbf{x},\mathbf{g}_k> \tilde{\mathbf{g}}_k$$

  is called a frame operator.
  - $\sum_{k\in\mathcal{K}} | <\mathbf{x},\mathbf{g}_k> |^2 = \|\mathbf{T}\mathbf{x}\|^2 =< \mathbf{T}\mathbf{x},\mathbf{T}\mathbf{x} >$
  - $< \mathbf{T}\mathbf{x},\mathbf{T}\mathbf{x} >=< \mathbf{T}^H\mathbf{T}\mathbf{x},\mathbf{x} >=< \mathbb{S}\mathbf{x},\mathbf{x} >$
- That is, $A\|\mathbf{x}\|^2 \leq < \mathbb{S}\mathbf{x},\mathbf{x} > \leq B\|\mathbf{x}\|^2$.

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Fourier Transform

- A signal $x(t) \in \mathcal{L}_2$ can be represented as a sum of complex exponentials, provided if it is band-limited.
- That is, $\mathbf{x} = \mathbf{F}^H\mathbf{F}\mathbf{x}$ where $\mathbf{F}$ forms a frame. In fact, it is a tight frame.
- $x(t) = \sum_k < x(t), f_k(t) > \tilde{f}_k(t)$.
- Similarly, all wavelet expansions make a frame.

Nyquist-Shannon Sampling Theorem

- Any bandlimited signal $x(t)$ with signal bandwidth W can be reconstructed from its samples $x[nT_s]$, if $T_s \leq \frac{1}{2W}$
- Here, the analysis function $\mathbf{g}_k$ is $Sinc(t - kT_s)$ which forms a frame if $T_s \leq \frac{1}{2W}$.
- Interestingly, it is a self-dual frame. That is, bi-orthonormal function of $Sinc(t - nT_s)$ is same as the analysis function.
- Note that, $Sinc(t - nT_s)$ forms an orthonormal basis, if sampling period condition is satisfied.

Sampling Theorem Continued ...

$Sinc(t - nT_s)$ forms an orthonormal basis, if sampling period condition is satisfied.

Proof:

- Since $X(f)$ is periodic in 2π, it has a Fourier series expansion. $X(f) = \sum_n x(nT_s)e^{-j\frac{2\pi fn}{f_s}}$
- The dual basis set $\{e^{-j2\pi\Omega n}, n = 1, 2, \ldots, 2W\}$ must be complete, in order to represent $X(f)$ uniquely.
- This is possible, if and only if $T_s < \frac{1}{2W}$.

Compressive Sensing

- The measurement kernel Φ used for observations $\mathbf{Y} = \Phi\mathbf{x}$, must satisfy the RIP conditions:

$$(1-\delta)\|\mathbf{x}\|^2 \leq \|\Phi\mathbf{x}\|^2 \leq (1+\delta)\|\mathbf{x}\|^2$$

- This means, the rows of Φ forms almost a tight frame.
- The condition on the coherence makes sure that, the data can be recovered reliably.
- The non-random Φ can be used to compute the canonical dual frame and use it to reduce the complexity in reconstruction.

References

1. V.I. Morgenshtern and H. Bolcskei,"*A Short Course on Frame Theory*", Arxiv:1104.4300V1 [cs.IT], 21 Apr 2011.
2. J. Kovacevic and A. Chebira,"*An Introduction to Frames*", Foundations and Trends in Signal Processing, Vol. 2, No. 1, pp. 1-94, NOW, 2008