# Analytical Characterization of Uncertainty in the Localization of a Sensor Node

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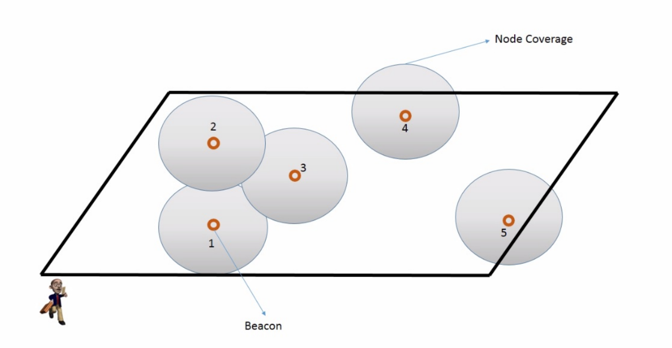
# Outline

# The Problem Setup

- A fixed number of beacon nodes (transmitters) deployed uniformly at random over an area
- Each beacon *covers* a circular area of radius $r$ around it (node coverage)
- The expected area covered is a function of $r$ and $n$
- A region in which every point is covered by at least $k$ beacons is said to be $k-$covered and its area is called the $k-$coverage

- A sensor (receiver) in the field of the beacons has a power vector associated with the location of its placement
- Each entry of the power vector is a binary value corresponding to the power received from a beacon
- Given a power vector, many locations over the area may be associated with the same vector, leading to uncertainty in the location of the sensor




## What We Want...

An mathematical expression relating the average uncertainty ($U_{avg}$) and average whitespace ($W_{avg}$) (both measured as a fraction of the total area) to $r$ and $n$, i.e.,

$$U_{avg} = f(r, n) \quad f = ?$$
$$W_{avg} = g(r, n) \quad g = ?$$

# Expected Coverage of a Single Beacon

Theorem 1 :

*If a beacon with transmission radius r is deployed uniformly at random in an $l \times b$ rectangular area ($r \leq min(l, b)/2$), its expected coverage is*

$$\mathbb{E}\left[A_{cov_{①}}\right] = \frac{\frac{r^4}{2} - \frac{4}{3}lr^3 - \frac{4}{3}br^3 + \pi r^2 lb}{lb}$$

*where $A_{cov_{①}}$ is a random variable representing the node coverage of a beacon and the expectation is computed over all possible locations of a beacon*

# Average Whitespace as a Function of $r$ and $n$

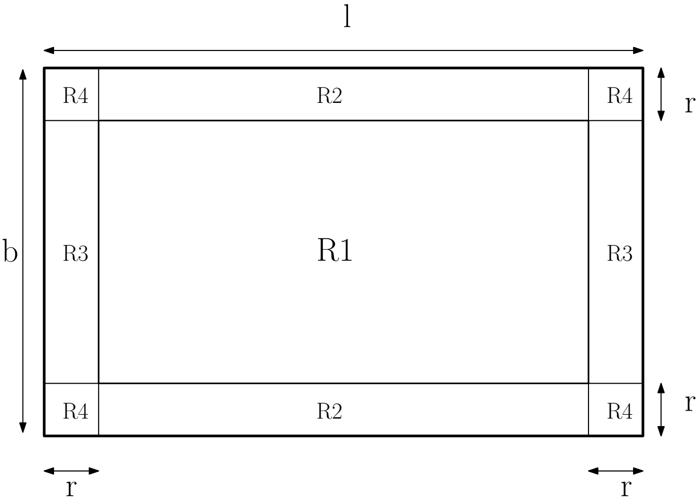
Theorem 2 :

*When $n$ beacons each with transmission radius $r$ are deployed uniformly at random in an $l \times b$ rectangular area ($r \leq \min(l, b)/2$), the average whitespace is*

$$W_{avg} = \left[1 - \left(\frac{\frac{r^4}{2} - \frac{4}{3}lr^3 - \frac{4}{3}br^3 + \pi r^2 lb}{l^2 b^2}\right)\right]^n = \left[1 - \frac{\mathbb{E}\left[A_{cov_{①}}\right]}{lb}\right]^n$$

*where $\mathbb{E}\left[A_{cov_{①}}\right]$ is the average area covered by a beacon and the expectation is computed over all possible locations of a beacon*

# Proof of Theorem 1




An $l \times b$ rectangular region over which the average area covered by a beacon deployed uniformly at random is to be computed

# Proof of Theorem 1

- Let $A_{cov_{①}}$ denote the area covered by a beacon of transmission radius $r$
- $A_{cov_{①}}$ is a random variable since its value depends on the location of the beacon
- Under the assumption that each beacon is deployed uniformly at random, the pdf of the beacon location $(x, y)$ is given by

$$f_{X,Y}(x,y) = \frac{1}{lb},\ 0 \le x \le l,\ 0 \le y \le b.$$

- The expected value of $A_{cov_{①}}$ can be written as

$$\mathbb{E}\left[A_{cov_{①}}\right] = \int_{x=0}^{l}\int_{y=0}^{b} a_{cov_{①}} f_{X,Y}(x,y)dxdy$$

# Proof of Theorem 1

- For the ease of analysis, the rectangular region is partitioned into 4 sub-regions - $R1$, $R2$, $R3$ and $R4$. The beacon can be deployed in one of the four regions
- The average area covered by a beacon may then be expressed as

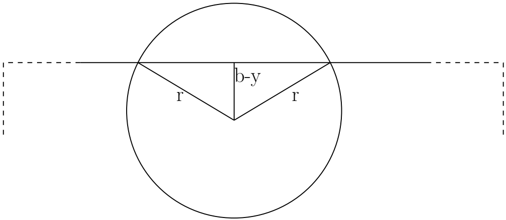
$$\mathbb{E}\left[A_{cov_{①}}\right] = \sum_{i=1}^{4} \mathbb{E}\left[A_{cov_{①}}\right]_{Ri} p(Ri)$$

  where $\mathbb{E}\left[A_{cov_{①}}\right]_{Ri}$ is the average area covered by a beacon deployed uniformly at random in sub-region $Ri$ and $p(Ri)$ is the probability that a beacon is deployed in sub-region $Ri$
- In sub-region $R1$,

$$a_{cov_{①}} = \pi r^2 = \mathbb{E}\left[A_{cov_{①}}\right]_{R1}$$

# Proof of Theorem 1

- In sub-region $R2$,



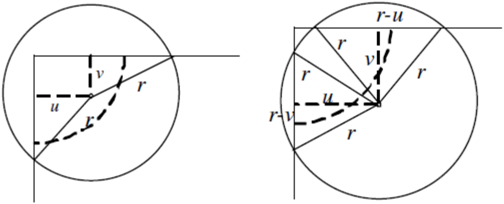

$$a_{cov_{①}} = (b-y)\sqrt{r^2-(b-y)^2} + r^2\left(\pi - cos^{-1}\left(\frac{b-y}{r}\right)\right)$$

$$\mathbb{E}\left[A_{cov_{①}}\right]_{R2} = r^2\left(\pi - \frac{2}{3}\right)$$

- Similar analysis yields the same value for $\mathbb{E}\left[A_{cov_{①}}\right]_{R3}$ as well

# Proof of Theorem 1

- In sub-region $R4$, there arise two scenarios




Accounting for both the scenarios depicted above,

$$\mathbb{E}\left[A_{cov_{①}}\right]_{R4} = r^2\left(\pi - \frac{29}{24}\right)$$

# Proof of Theorem 1

- $p(R1)$ is given as follows:

$$p(R1) = \frac{\text{Size of sub-region R1}}{\text{Total area of the region}} = \frac{(l-2r)(b-2r)}{lb}$$

- Similarly,

$$p(R2) = \frac{2r\,(l-2r)}{lb},\ p(R3) = \frac{2r\,(b-2r)}{lb},\ p(R4) = \frac{4r^2}{lb}$$

- Substituting all the values yields

$$\mathbb{E}\left[A_{cov_{①}}\right] = \frac{\frac{r^4}{2} - \frac{4}{3}lr^3 - \frac{4}{3}br^3 + \pi r^2 lb}{lb} \qquad \blacksquare$$

# Proof of Theorem 2

- Let $E_1,\ E_2,\ \cdots,\ E_n$ denote the $n$ independent events that a location belongs to the area covered by beacon 1, beacon 2, $\cdots$, beacon $n$ respectively. From Theorem 1, we may write

$$p\left(E_i\right) = \frac{\mathbb{E}\left[A_{cov_{①}}\right]}{lb},\ 1 \leq i \leq n$$

- Let $W$ denote the event that a location belongs to whitespace (no-coverage region) associated with the deployment of $n$ beacons. Clearly,

$$W = \left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} \left(E_i\right)^c$$

- The probability of occurrence of event $W$ is

$$p(W) = \prod_{i=1}^{n} p\left[\left(E_i\right)^c\right]$$

# Proof of Theorem 2

$$p(W) = \left(1 - \frac{\mathbb{E}\left[A_{cov_{①}}\right]}{lb}\right)^n$$

- $p(W)$ quantifies the average whitespace as a fraction of the total area. Letting $W_{avg}$ denote this, we get

$$W_{avg} = \left(1 - \frac{\mathbb{E}\left[A_{cov_{①}}\right]}{lb}\right)^n$$
$$= \left[1 - \left(\frac{\frac{r^4}{2} - \frac{4}{3}lr^3 - \frac{4}{3}br^3 + \pi r^2 lb}{l^2 b^2}\right)\right]^n \quad \blacksquare$$

# Uncertainty in the Localization of a Sensor

- Every location in the area is associated with a power vector
- Given a binary-valued power vector, it is difficult to associate a unique location with it since many locations may be associated with the same vector
- Let $A_{cov_{Ⓚ}}$ be a random variable denoting the area of a region in which every location is covered by exactly $k$ out of $n$ beacons. The fraction of the total area that is exactly $k-$covered can be expressed as

$$f_{Ⓚ} = \frac{\mathbb{E}\left[A_{cov_{Ⓚ}}\right]}{lb}$$

# Uncertainty in the Localization of a Sensor

- There exist $\binom{n}{k}$ binary-valued power vectors having $k$ ones. The fraction of the total area associated with one such vector is

$$f_{\textcircled{k}_{\binom{n}{k}}} = \frac{f_{\textcircled{k}}}{\binom{n}{k}}$$

- The average uncertainty (as a fraction of the total area) in localization associated with a power vector with exactly can be expressed as

$$U_{avg} = \sum_{k=0}^{n} \binom{n}{k} \left( f_{\textcircled{k}_{\binom{n}{k}}} \right)^2$$

# Uncertainty in the Localization of a Sensor

- To analytically characterize $U_{avg}$, it is necessary to express $\mathbb{E}\left[A_{cov_{Ⓚ}}\right]$ in terms of $r$ and $n$
- Letting $C_n^k$ denote the area covered by at least $k$ out of $n$ beacons, we may write

$$\mathbb{E}\left[A_{cov_{Ⓚ}}\right] = \mathbb{E}\left[C_n^k\right] - \mathbb{E}\left[C_n^{k+1}\right]$$

- Suppose that there are $i$ beacons deployed and the area of the $j-$covered region is $C_i^j$. If the $(i+1)^{\text{th}}$ beacon adds an extra area $X_{i+1}^j$ to the $j-$covered region, the new size of $j-$covered region will be

$$C_{i+1}^j = C_i^j + X_{i+1}^j$$

# Uncertainty in the Localization of a Sensor

- If $F_{i+1}^{j}$ denotes the fraction of the extra area contributed by the addition of the $(i+1)^{\text{th}}$ beacon, it is expected to be

$$\mathbb{E}\left[F_{i+1}^{j}\right] = \frac{\mathbb{E}\left[C_i^j\right] - \mathbb{E}\left[C_i^{j+1}\right]}{lb}$$

- An recursive formula[1] shown below can be used to evaluate $\mathbb{E}\left[C_i^j\right]$

$$\mathbb{E}\left[C_i^j\right] = p\mathbb{E}\left[C_{i-1}^{j-1}\right] + (1-p)\mathbb{E}\left[C_{i-1}^{j}\right]$$

where $p = \frac{\mathbb{E}\left[A_{cov}\textcircled{1}\right]}{lb}$

[1] Result from 'Expected k-coverage in Wireless Sensor Networks', Li-Hsing Yen et al.

# Simulation Results

- $l = 25$m, $b = 25$m
- A $100 \times 100$ grid assumed to evaluate coverage area
- Number of random deployment experiments over which averaging is performed $= 10000$

# Average Whitespace vs. $r$

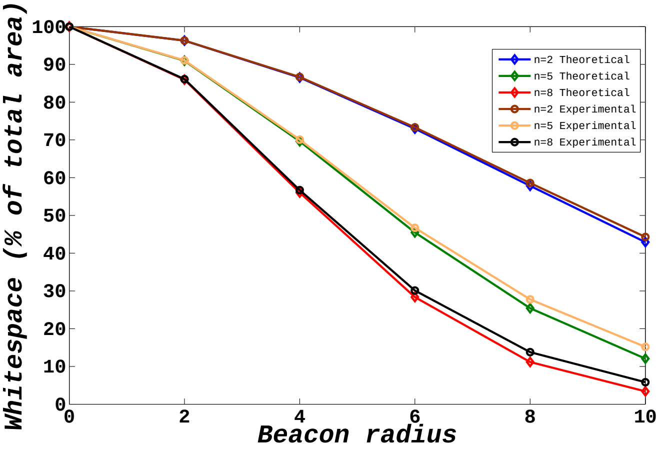



Figure: Average whitespace (as a percentage of the total area) as a function of beacon radius. The mean absolute percentage error between theoretical and experimental values = 5.98%

# Average Uncertainty vs. $r$

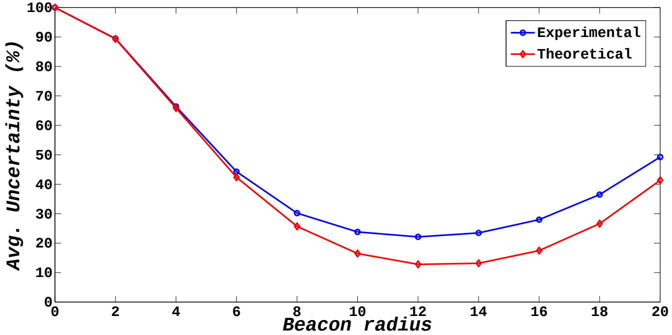



Figure: Average uncertainty (as a percentage of the total area) as a function of beacon radius for $n = 3$ and $r = 0 : 2 : 10$m.

# Summary

- Average uncertainty in the localization of a sensor placed in a field of beacons and available whitespace are functions of beacon radius ($r$) and number of beacons ($n$)
- The expected coverage of a single beacon deployed in a region of finite dimensions is lesser than its *node coverage*
- When two or more beacons are deployed uniformly at random, the average size of whitespace region is a polynomial decreasing function of beacon radius
- The average uncertainty in the association of a unique location in the region, given a vector of power values, attains a minimum for a certain value of $r$, for a given value of $n$.