Construction of Unimodular Tight Frames Via Majorization-Minimization Methods

Ramu Naidu R

IISc, SPC lab

November 26, 2016

Organization

- Basics of Frame Theory
- Basics of Compressed Sensing
- The Majorization-Minimization (MM) method
- Construction of Unimodular Tight Frames (UNTF) via MM methods
- Simulation results

Basics of Frame Theory

Definition

A family of vectors $\Psi = \{\psi_i\}_{i=1}^{M}$ in $\mathbb{C}^m$ is called a frame for $\mathbb{C}^m$, if there exist constants $0 < A \leq B < \infty$ such that

$$A \|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \psi_i \rangle|^2 \leq B \|x\|^2, \forall x \in \mathbb{C}^m, \quad (1)$$

where A, B are called the lower and upper frame bounds respectively.

- The matrix $\Psi_{m \times M} = [\psi_1, \ldots, \psi_M]$ with ψ_i as columns, is known as the frame synthesis operator
- The optimal frame bounds A and B are the least and greatest eigenvalues of $\Psi\Psi^*$

Construction of UNTFs via MM methods

Basics of Frame Theory

- If $A = B$, then $\{\psi_i\}_{i=1}^{M}$ is an $A-$tight frame (TF)
- If $A = B = 1$, then it is a Parseval frame
- The tightness condition of a frame Ψ implies that the rows of Ψ are orthogonal and have equal norm $\sqrt{A}$
- If there exist a constant d such that $|\langle\psi_i, \psi_j\rangle| = d$, for $1 \leq i < j \leq M$, then it is an equiangular frame
- If there exits a constant c such that $\|\psi_i\| = c$ for all $i = 1, 2, \ldots, M$ then $\{\psi_i\}_{i=1}^{M}$ is an equal norm frame
- If $c = 1$ then it is a unit-norm frame
- If a frame is unitnorm TF and equiangular, then it is called equiangular tight frame(ETF)
- $\{\langle x, \psi_i\rangle\}_{i=1}^{M}$ is called the frame coefficients of $x \in \mathbb{C}^m$ with respect to frame $\{\psi_i\}_{i=1}^{M}$

Construction of UNTFs via MM methods

Basics of Frame Theory

- Tight frames provide following Parseval-like decompositions:

$$x = \frac{1}{A}\sum_{i=1}^{M} \langle x, \psi_i \rangle \psi_i, \forall x \in \mathbb{C}^m. \tag{2}$$

- Any unitnorm $A-$tight frame satisfies the following condition:

$$M = \sum_{i=1}^{M} \|\psi_i\|_2^2 = Trace(\Psi^*\Psi) = Trace(\Psi\Psi^*) = Am. \tag{3}$$

- From the above equation, one can conclude that for an unitnorm $A-$tight frame, the tightness parameter $A = \frac{M}{m}$
- Unitnorm tight frames, have been used in the construction of signature sequences in CDMA systems

Basics of Frame Theory

- A tight frame is said to be Unimodular tight frame (UTF) if $|\psi_{ij}| = 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, M$
- If $\Psi_{m\times M}$ is a unimodular tight frame then $\frac{1}{\sqrt{m}}\Psi$ is a unitnorm tight frame

Definition

The mutual-coherence $\mu(\Psi)$ of a given frame Ψ is the largest absolute inner-product between different normalized frame columns of Ψ, that is,

$$\mu(\Psi) = \max_{1\leq i,j\leq M,\ i\neq j} \frac{|\ \psi_i^T \psi_j|}{\|\psi_i\|_2 \|\psi_j\|_2}. \tag{4}$$

- We define frames that have a low mutual coherence value to be incoherent

Construction of UNTFs via MM methods

Basics of Frame Theory

- The Welch bound for any arbitrary frame is as follows:

$$\mu(\Psi) \geq \sqrt{\frac{M-m}{m(M-1)}} \tag{5}$$

- Equiangular tight frames are the unit norm ensembles that achieve equality in the above Welch bound
- The construction of unit norm tight frames and equiangular tight frames has been proven notoriously difficult
- In the construction of unit norm tight frames, the frame potential,

$$FP(\Psi) = \sum_{i=1}^{M}\sum_{j=1}^{M} |\langle \psi_i, \psi_j \rangle|^2, \tag{6}$$

is a useful tool

Construction of UNTFs via MM methods

Basics of Frame Theory

- In particular, if $\mathcal{F}$ is the family of frames with lower frame bound A then the A-tight frames are the minimizers of the frame potential over $\mathcal{F}$
- That is, UNTFs are exactly the minimizers of a frame potential
- Any UNTF with $M = m + 1$ vectors is an equiangular tight frame
- In this paper, we construct incoherent unimodular tight frames using majorization-minimization methods
- Based on recent theoretical results, we employ these frames in compressed sensing to improve reconstruction of sparse signals

Basics of Compressed Sensing

- A vector $x \in \mathbb{R}^M$ is $k-$**sparse** if it has k nonzero coordinates. That is, $\|x\|_0 := |\{i \mid x_i \neq 0\}| = k < M$
- One of the central problems in CS is that of reconstructing an unknown sparse vector $x \in \mathbb{R}^M$ from the linear measurements $y' = (\langle x, \psi_1 \rangle, \ldots, \langle x, \psi_m \rangle) \in \mathbb{R}^m$
- One can recover sparse x from its linear measurements by solving the following optimization problem:

$$P_0 : \min_x \|x\|_0 \text{ subject to } \quad \Psi x = y \qquad (7)$$

- This l_0-minimization problem is computationally not tractable[a] in general

[a] Simon Foucart and Holger Rauhut, "A Mathematical Introduction to Compressive Sensing," Birkhauser, Baseln, 2013.

On the solvability of P_0 problem

- There have been attempts to repose or solve P_0 problem via greedy and convex relaxation methods
- D.Donoho et.al. posed an equivalent of this problem as

$$P_1 : \min_x \|x\|_1 \text{ subject to } \quad b = \Psi x \tag{8}$$

- The general question of CS is: "when do both problems (7) and (8) admit same solution ?"

Theorem

An arbitrary $k-$sparse signal x can be uniquely recovered from $y = \Psi x$ as a solution to P_1 problem provided

$$k < \frac{1}{2}\left(1 + \frac{1}{\mu_\Psi}\right). \tag{9}$$

The Majorization-Minimization (MM) method

- Suppose we want to minimize $f(x)$ over $\mathcal{X} \subset \mathbb{C}^m$
- The MM approach optimizes a sequence of approximate objective functions $g(x, x^{(k)})$ that majorize $f(x)$
- The function $g(x, x^{(k)})$ is said be majorized function of $f(x)$ at the point $x^{(k)}$ if

$$f(x) \leq g(x, x^{(k)}), \forall x \in \mathcal{X}, \text{ and } f(x^{(k)}) = g(x^{(k)}, x^{(k)}) \quad (10)$$

- The MM algorithm corresponding to this majorization function g, starts with a random feasible point x^0 and produces a sequence $\{x^{(k)}\}$ according to the following update rule

$$x^{(k+1)} = \arg\min_{x \in \mathcal{X}} g(x, x^{(k)}) \quad (11)$$

- The above iterative scheme decreases the value of f monotonically in each iteration

Construction of Unimodular Tight Frames Via MM Methods

- By solving the following optimization problem one can generate UTFs,

$$\arg\min_{\{\psi_l\}_{l=1}^{M}} \sum_{i=1}^{M}\sum_{j=1}^{M} |\langle \psi_i, \psi_j \rangle|^2 \text{ s.t } |\psi_{ij}| = 1; \forall i \text{ and } j. \quad (12)$$

- Let's assume $\Psi = [\psi_1^T, \ldots, \psi_M^T]^T$, then we have

$$\psi_l = S_l \Psi, \quad (13)$$

where S_l is an $m \times mM$ block selection matrix defined as

$$S_l = [0_{m\times(l-1)m}, I_m, 0_{m\times(M-l)m}]$$

- We define the inner-product between any two vectors of $\mathbb{C}^m$ as $\langle \psi_i, \psi_j \rangle = \psi_j^H \psi_i$

Construction of Unimodular Tight Frames Via MM Methods

- From (13), we have,

$$\langle \psi_i, \psi_j \rangle = \Psi^H S_j^H S_i \Psi, \tag{14}$$

- Which implies,

$$\begin{aligned} |\langle \psi_i, \psi_j \rangle|^2 &= |\Psi^H S_j^H S_i \Psi|^2 \\ &= |tr(\Psi^H S_j^H S_i \Psi)|^2 \\ &= |vec(\Psi \Psi^H)^H vec(S_j^H S_i)|^2, \end{aligned} \tag{15}$$

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- By using (15), the minimization problem in (12) can be written as

$$\underset{\Psi \in \mathbb{C}^{mM}}{\arg\min} \quad vec(\Psi\Psi^H)^H L vec(\Psi\Psi^H) \qquad (16)$$
$$\text{subject to } |\psi_i| = 1; \forall i = 1, \ldots, mM,$$

where

$$L = \sum_{i=1}^{M} \sum_{j=1}^{M} vec(S_j^H S_i) vec(S_j^H S_i)^H. \qquad (17)$$

Lemma

Let P be an $m \times m$ Hermitian matrix and Q be another $m \times m$ Hermitian matrix such that $Q \succeq P$. Then for any point $X_0 \in \mathbb{C}^m$, the quadratic function $X^H P X$ is majorized by $X^H Q X + 2Re(X^H (P - Q) X_0) + X_0^H (Q - P) X_0$ at x_0.

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

Lemma

Let $P_{m\times m}$ be a real symmetric non-negative matrix. Then the problem

$$\min_{b} b^T 1_m \text{ subject to } Diag(b) \succeq P$$

admits the optimal solution, which is $b^* = P1_m$, where $Diag(.)$ is a diagonal matrix formed with the vector $(.)$, as its principal diagonal.

- It is easy to see that L in equation (17) is a nonnegative real symmetric matrix and by using above lemma it can be shown that $L \preceq Diag(b)$, where $b = L1_{m^2 M^2}$.

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- Therefore for a given $\Psi^{(l)}$ at iteration l by using first lemma the majorized function for the objective function in equation (16) at $\Psi^{(l)}$ as follows:

$$\begin{aligned} g_1(\Psi, \Psi^{(l)}) \\ &= vec(\Psi\Psi^H)^H Diag(b) vec(\Psi\Psi^H) \\ &+2Re(vec(\Psi\Psi^H)^H (L - Diag(b)) vec(\Psi^{(l)}\Psi^{(l)^H})) \\ &+vec(\Psi^{(l)}\Psi^{(l)^H})^H (Diag(b) - L) vec(\Psi^{(l)}\Psi^{(l)^H}) \end{aligned} \tag{18}$$

- After ignoring the constant terms the majorized problem of equation (16) is given by

$$\begin{aligned} \min_{\Psi} \quad & Re(vec(\Psi\Psi^H)^H (L - Diag(b)) vec(\Psi^{(l)}\Psi^{(l)^H})) \\ & \text{subject to } |\psi_i| = 1; \forall i = 1, \dots, mM. \end{aligned} \tag{19}$$

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- After some simplifications the minimization problem in equation (19) cab be written as follows:

$$\min_{\Psi} \quad \Psi^H \left(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H})) \right) \Psi \qquad (20)$$
$$\text{subject to } |\psi_i| = 1; \forall i = 1, \ldots, mM,$$

where

$$R = 1_{M \times M} \otimes mI_m \text{ and } S = \sum_{i=1}^{M} \sum_{j=1}^{M} \left\langle \psi_j^{(l)}, \psi_i^{(l)} \right\rangle (S_j^H S_i) \quad (21)$$

- We majorize the objective function at $\Psi^{(l)}$ to further simplify the problem at every iteration

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- To construct a majorization function for the objective function in (20), we need to find a matrix Q such that $(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H}))) \preceq Q$ and the obvious choice may be $Q = \lambda_{max}(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H})))I$. We have the following:

$$\begin{aligned} \lambda_{max}(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H}))) &\leq \lambda_{max}(S) - \lambda_{min}(R), \\ &\leq \|S\|_\infty - \lambda_{min}(R) \end{aligned} \tag{22}$$

- Since $M \geq m$ and the eigenvalues of $A \otimes B$ are the product of eigenvalues of A and B, we have $\lambda_{min}(R) = 0$. Therefore,

$$\lambda_{max}(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H}))) \leq \|S\|_\infty \tag{23}$$

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- Now by choosing $Q = \|S\|_\infty I$ in first lemma, the objective function in equation (20) is majorized by

$$
\begin{aligned}
& g_2(\Psi, \Psi^{(l)}) \\
&\qquad = \|S\|_\infty \Psi^H \Psi \\
&\qquad + 2Re\left(\Psi^H \left(S - (R \odot (\Psi^{(l)} \Psi^{(l)^H})) - \|S\|_\infty I \right) \Psi^{(l)} \right) \\
&\qquad + \Psi^{(l)^H} \left(\|S\|_\infty I - S + (R \odot (\Psi^{(l)} \Psi^{(l)^H})) \right) \Psi^{(l)}. \quad (24)
\end{aligned}
$$

- After ignoring the constant terms, the majorized problem of (20) is given by

Construction of UTFs via MM methods

Construction of Unimodular Tight Frames Via MM Methods

- $$\min_{\Psi} \quad Re(\Psi^H \Phi) \tag{25}$$
  $$\text{subject to } |\psi_i| = 1; \forall i = 1, \ldots, mM,$$
  where $\Phi = (S - (R \odot (\Psi^{(l)} \Psi^{(l)^H})))\Psi^{(l)} - \|S\|_\infty \Psi^{(l)}$
- It is clear that the minimization problem in equation (25) is separable in the elements of Ψ and the solution is as follows:
  $$\psi_i = e^{j arg(-\phi_i)}; \forall i = 1, \ldots, mM \tag{26}$$

Construction of UTFs via MM methods

Simulation results

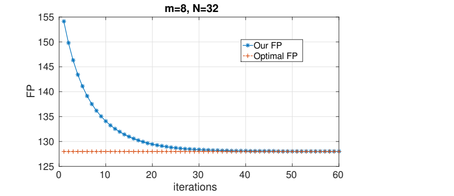

Figure : Comparison of the reconstruction performances of the synthesized matrices and complex Gaussian random matrices when the matrices are of size (a) 16×64 (top plot). These plot indicate that the matrices constructed from our approach show superior performance for some sparsity levels, while for other levels both matrices result in the same performance. The x and y axes in both plots refer respectively to the sparsity level and the success rate (in % terms).

References

- Vivek K. Goyal, Jelena Kovaevi, and Jonathan A. Kelner, “Quantized frame expansions with erasures,” Applied and Computational Harmonic Analysis, vol. 10, no. 3, pp. 203 - 233, 2001.
- Peter G. Casazza and Jelena Kova, “Equal-norm tight frames with erasures,” Advances in Computational Mathematics, vol. 18, no. 2, pp. 387-430, 2003.
- Y. C. Eldar and G. D. Forney, “Optimal tight frames and quantum measurement,” IEEE Transactions on Information Theory, vol. 48, no.3, pp. 599-610, Mar 2002.
- Roderick B Holmes and Vern I Paulsen, “Optimal frames for erasures,” Linear Algebra and its Applications, vol. 377, pp. 31 - 51, 2004.
- R. Gribonval and M. Nielsen, “Sparse representations in unions of bases,” IEEE Transactions on Information Theory, vol. 49, no. 12, pp. 3320-3325, Dec 2003.

References

- Thomas Strohmer and Robert W Heath, "Grassmannian frames with applications to coding and communication," Applied and Computational Harmonic Analysis, vol. 14, no. 3, pp. 257 - 275, 2003.
- John J. Benedetto and Matthew Fickus, "Finite normalized tight frames," Advances in Computational Mathematics, vol. 18, no. 2, pp. 357-385, 2003.
- S. Foucart and H. Rauhut, "A mathematical introduction to compressive sensing," Birkhauser, Baseln, 2013.
- E. J. Candes and T. Tao, "Decoding by linear programming," IEEE Transactions on Information Theory, vol. 51, no. 12, pp. 4203-4215, Dec 2005.
- P. G. Casazza and G. Kutyniok, "Finite frames: Theory and applications," Birkhauser Boston, Inc., Boston, MA, 2012.

References

- E. Tsiligianni, L. P. Kondi, and A. K. Katsaggelos, "Preconditioning for underdetermined linear systems with sparse solutions," IEEE Signal Processing Letters, vol. 22, no. 9, pp. 1239-1243, Sept 2015.
- J. Song, P. Babu, and D. P. Palomar, "Sequence set design with good correlation properties via majorization-minimization," IEEE Transactions on Signal Processing, vol. 64, no. 11, pp. 2866-2879, June 2016.
- David R Hunter and Kenneth Lange, "A tutorial on mm algorithms," The American Statistician, vol. 58, no. 1, pp. 30-37, 2004.
- J. Song, P. Babu, and D. P. Palomar, "Optimization methods for designing sequences with low autocorrelation sidelobes," IEEE Transactions on Signal Processing, vol. 63, no. 15, pp. 3998-4009, Aug 2015.

Thank you