Group Discussion

Maximum Hands-Off Control: A Paradigm of Control Effort Minimization

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Hands-off control

- A hands-off control is defined as a control that has a short support per unit time.
- The maximum hands-off control is the minimum support (or sparsest) per unit time among all controls that achieve control objectives.
- The energy (or L^2 -norm) of a control signal can be minimized to prevent engine overheating or to reduce transmission cost by means of a standard LQ (linear quadratic) control problem.
- The minimum fuel control in which the total expenditure of fuel is minimized with the L^1 norm of the control.

Motivation

- In some situations, the control effort can be dramatically reduced by holding the control value exactly zero over a time interval. We call such control a hands-off control.
- A motivation for hands-off control is a stop-start system in automobiles. It is a hands-off control; it automatically shuts down the engine to avoid it idling for long periods of time.
- This strategy is also used in electric/hybrid vehicles, the internal combustion engine is stopped when the vehicle is at a stop or the speed is lower than a preset threshold, and the electric motor is alternatively used.

Usefulness

- We can reduce CO or CO2 emissions as well as fuel consumption. Hands-off control also has potential for solving environmental problems.
- In railway vehicles, hands-off control, called coasting, is used to reduce energy consumption.
- Hands-off control is desirable for networked and embedded systems since the communication channel is not used during a period of zero-valued control. This property is advantageous in particular for wireless communications and networked control systems.

Objective

- For finite horizon continuous-time control, it is to show the equivalence between the maximum hands-off control and L^1 -optimal control under a uniqueness assumption called normality.
- This result rationalizes the use of L^1 optimality in computing a maximum hands-off control.
- The same result is obtained for discrete-time hands-off control and an L^1/L^2-optimal control to obtain a smooth hands-off control.

Sparsity

- For a continuous-time signal $u(t)$ over a time interval $[0, T]$, we define its L^p norm with $p \in [1, \infty)$ by

$$\|u\|_p = \left(\int_0^T |u(t)|^p dt \right)^{\frac{1}{p}} \tag{1}$$

- $L^p[0, T] = \{u(t) : \|u\|_p < \infty\}$
- $supp(u) = \overline{\{t \in [0, T] : u(t) = 0\}}$
- $\|u\|_0 := m_L(supp(u))$, where m_L is the Lebesgue measure on $\mathbb{R}$.
- The L^0 norm is not a norm since it fails to satisfy the positive homogeneity property, that is, for any non-zero scalar α such that $|\alpha| \neq 1$, we have $\|\alpha u\|_0 = \|u\|_0 \neq |\alpha| \|u\|_0, \quad \forall u \neq 0$.
- The notation $\| \cdot \|_0$ may be however justified from the fact that if $u \in L^1[0, T]$, then $\|u\|_p < \infty$ for any $p \in (0, 1)$ and $\lim_{p \to 0} \|u\|_p^p = \|u\|_0$. which can be proved by using Lebesgues monotone convergence theorem.

Maximum hand-off control problem

Definition

Sparsity Rate: For a measurable function u on $[0, T]$, $T > 0$, the sparsity rate is defined by

$$R_T(u) := \frac{1}{T}\|u\|_0. \tag{2}$$

- For any measurable function u, $0 \leq R_T(u) \leq 1$. if $R_T(u) << 1$, then we say that u is sparse.
- The control objective is, roughly speaking, to design a control u which is as sparse as possible, whilst satisfying performance criteria.

Problem formulation

To formulate the control problem, we consider nonlinear multi-input plant models of the form

$$\frac{dx(t)}{dt} = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t), \quad t \in [0, T] \tag{3}$$

where

- $x(t) \in \mathbb{R}^n$ is the state,
- $u_1, \dots, u_m$ are the scalar control inputs,
- f and g_i are functions on $\mathbb{R}^n$. We assume that $f(x), g_i(x)$, and their Jacobians $f'(x), g_i'(x)$ are continuous.
- We use the vector representation $u := [u_1, \dots, u_m]$.

Admissible

- The control $u(t) : t \in [0, T]$ is chosen to drive the state $x(t)$ from a given initial state
$$x(0) = \xi. \tag{4}$$
to the origin at a fixed final time $T > 0$, that is
$$x(T) = 0. \tag{5}$$
- Also, the components of the control $u(t)$ are constrained in magnitude by
$$\max_i |u_i(t)| \leq 1 \tag{6}$$
for all $t \in [0, T]$
- We call a control $u(t) : t \in [0, T] \in L^1[0, T]$ admissible if it satisfies (6) for all $t \in [0, T]$, and the resultant state $x(t)$ from (3) satisfies boundary conditions (4) and (5). We denote by $\mathbf{U}(T, \xi)$ the set of all admissible controls.

Minimum-Time Control and Reachable Set

To consider control in $\mathbf{U}(T,\xi)$, it is necessary that $\mathbf{U}(T,\xi) \neq \emptyset$. This property is basically related to the minimum time control formulated as follows.

Problem 2(Minimum-Time Control):

Find a control $u \in L^1[0, T]$ that satisfies (6), and drives x from initial state $\xi \in \mathbb{R}^n$, to the origin 0 in minimum time.

Let $T^*(\xi)$ denote the minimum time (or the value function) of Problem 2.

Definition

(Reachable Set): We define the reachable set at time $t \in [0,\infty)$ by

$$\mathbf{R}(t) := \{\xi \in \mathbb{R}^n : T^*(\xi) \leq t\}. \tag{7}$$

and the reachability set

$$\mathbf{R} := \bigcup_{t \geq 0} \mathbf{R}(t). \tag{8}$$

Problem formulation

To guarantee that $\mathbf{U}(T, \xi)$ is non-empty, we introduce the standing assumptions:

- $\xi \in \mathbf{R}$
- $T > T^*(\xi)$.

Now let us formulate our control problem. The maximum hands-off control is a control that is the sparsest among all admissible controls in $\mathbf{U}(T, \xi)$. In other words, we try to find a control that maximizes the time interval over which the control $u(t)$ is exactly zero. We state the associated optimal control problem as follows.
Problem 4 (Maximum Hands-Off Control): Find an admissible control on $[0, T]$, $u \in \mathbf{U}(T, \xi)$, that minimizes the sum of sparsity rates

$$J_0(u) := \sum_{i=1}^{m} \lambda_i R_T(u_i) = \frac{1}{T}\lambda_i \|u_i\|_0, \tag{9}$$

where $\lambda_i > 0$ are given weights.

Convex Relaxation

Problem 6 (L^1-Optimal Control): Find an admissible control $u \in \mathbf{U}(T, \xi)$ on $[0, T]$ that minimizes

$$J_1(u) := \frac{1}{T}\lambda_i \|u_i\|_1 = \frac{1}{T}\sum_{i=1}^{m}\lambda_i \int_0^T |u(t)|dt, \qquad (10)$$

where $\lambda_i > 0$ are given weights.

The objective function (10) is convex in u and this control problem is much easier to solve than the maximum hands-off control problem

Review of L^1-Optimal Control

Let us first form the Hamiltonian function for the L^1-optimal control problem as

$$H(x, p, u) = \frac{1}{T}\sum_{i=1}^{m} \lambda_i \|u_i\|_1 + p^T(f(x) + \sum_{i=1}^{m} g(x)u_i)$$

(11)

where p is the costate (or adjoint) vector. Assume that $u^* = [u_1^*, \ldots, u_m^*]$ is an L^1-optimal control and x^* is the resultant state trajectory.

Optimal Conditions

- According to minimum principle, $\exists p^*$ such that the optimal control u^* satisfies

$$H(x^*(t), p^*(t), u^*(t)) \leq H(x^*(t), p^*(t), u(t)),$$

  $\forall t \in [0, T]$ and $\forall u \in \mathbf{u}(T, \xi)$.
- $\frac{dx^*(t)}{dt} = f(x^*(t)) + \sum_{i=1}^{m} g_i(x^*(t))u_i^*(t)$.
- $\frac{dp^*(t)}{dt} = -f'(x^*(t))^T p^*(t) - \sum_{i=1}^{m} u_i^*(t)g_i'(x^*(t))p^*(t)$.
- $x^*(0) = \xi$ and $x^*(T) = 0$

The minimizer $u^* = [u_1^*, \ldots, u_m^*]$ of the Hamiltonian is given by

- $u_i^*(t) = -D_{\frac{\lambda_i}{T}}(g_i(x^*(t))p^*(t)), \quad t \in [0, T]$

- $D_\lambda(\cdot) : \mathbb{R}^n \to [-1, 1]$ is the dead-zone function defined by

$$D_\lambda(w) = \begin{cases} -1 & w < -\lambda \\ 0 & -\lambda < w < \lambda \\ 1 & \lambda < w \\ a & w = -\lambda \\ b & w = \lambda \end{cases} \qquad \text{a} \in [-1, 0] \text{ and } b \in [0, 1]$$

Normality

- If $g_i(x^*(t))p^*(t)$ is equal to $\frac{-\lambda_i}{T}$ or $\frac{\lambda_i}{T}$ over a non-zero time interval, say $[t_1, t_2] \in [0, T]$, where $t_1 < t_2$, then the control u_i (and hence u) over $[t_1, t_2]$ cannot be uniquely determined by the minimum principle.
- The interval $[t_1, t_2]$ is called a singular interval, and a control problem that has at least one singular interval is called singular. If there is no singular interval, the problem is called normal.

Definition

(Normality) The L^1-optimal control problem stated in Problem 6 is said to be normal if the set

$$T_i = \{t \in [0, T] : |T\lambda_i^{-1} g_i(x^*(t))p^*(t)| = 1\}$$

is countable for $i = 1, \cdots, m$. If the problem is normal, the elements $t_1, t_2, \cdots \in T_i$ are called the switching times for the control $u_i(t)$.

Equivalence between L^0 and L^1

If the problem is normal, the components of the L^1−optimal control $u^*(t)$ are piecewise constant and ternary, taking values 1, -1 or 0 at almost all $t \in [0, T]$. This property, named bang-off-bang, is the key to relate the L^1-optimal control with the maximum hands-off control.

Theorem

Assume that the L^1-optimal control problem (Problem 6) is normal and has at least one solution. Let $\mathbf{U}_0^*$ and $\mathbf{U}_1^*$ be the sets of the optimal solutions of Problem 4 (maximum hands-off control problem) and Problem 6, respectively. Then we have $\mathbf{U}_0^* = \mathbf{U}_1^*$.

Proof

By assumption, $\mathbf{U}_1^* \neq \emptyset \Rightarrow \mathbf{U}(T, \xi) \neq \emptyset$. We first show that $\mathbf{U}_0^* \neq \emptyset$ and then prove that $\mathbf{U}_0^* = \mathbf{U}_1^*$.
For any $u \in \mathbf{U}(T, \xi)$, we have

$$J_1(u) = \frac{1}{T} \sum_{i=1}^{m} \lambda_i \int_0^T |u(t)| dt$$

$$= \frac{1}{T} \sum_{i=1}^{m} \lambda_i \int_{supp(u_i)} |u(t)| dt$$

$$\leq \frac{1}{T} \sum_{i=1}^{m} \lambda_i \int_{supp(u_i)} 1 dt = J_0(u) \qquad (13)$$

Now take an arbitrary $u_1^* \in \mathbf{U}_1^*$. Since the problem is normal by assumption, each control u_{1i}^* in u_1^* takes values $-1, 0, 1$, at almost all $t \in [0, T]$. This implies that

$$J_1(u_1^*) = \frac{1}{T} \sum_{i=1}^{m} \lambda_i \int_0^T |u_{1i}^*(t)| dt$$

$$\leq \frac{1}{T} \sum_{i=1}^{m} \lambda_i \int_{supp(u_{1i}^*)} 1 dt = J_0(u_1^*) \qquad (14)$$

From (13) and (14), u_1^* is a minimizer of J_0, that is $u_1^* \in \mathbf{U}_0^*$. Thus, $\mathbf{U}_0^* \neq \emptyset$ and $\mathbf{U}_1^* \subset \mathbf{U}_0^*$.

Conversely, let $u_0^* \in \mathbf{U}_0^* \subset \mathbf{U}(T,\xi)$. Take independently, $u_1^* \in \mathbf{U}_1^* \subset \mathbf{U}(T,\xi)$. From (14) and optimality of u_1^*, we have

$$J_0(u_1^*) = J_1(u_1^*) \leq J_1(u_0^*). \quad (15)$$

On the other hand, from (13) and optimality of u_0^*, we have

$$J_1(u_0^*) \leq J_0(u_0^*) = J_0(u_1^*). \quad (16)$$

It follows from (15) and (16) that $J_1(u_1^*) = J_1(u_0^*)$, and hence u_0^* achieves the minimum value of J_1. That is, $u_0^* \in \mathbf{U}_1^*$ and $\mathbf{U}_0^* \subset \mathbf{U}_1^*$.

References

M. Nagahara, D. E. Quevedo and D. Nesic, “Maximum Hands-Off Control: A Paradigm of Control Effort Minimization,” IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 61, NO. 3, MARCH 2016.

Thanks for your Patient Listening