Online Recovery of Correlated Sparse Signals using Multiple Measurement Vectors

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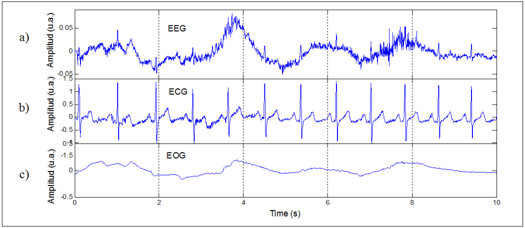
Agenda

- Recap: Sparse Bayesian Learning
- MMV Setup with AR model
- Two Online Recovery Schemes
  - Fixed Lag Smoothing Approach
  - Sawtooth Lag Smoothing Approach

Motivation

- Natural signals are known to be sparse and have significant structure
  - Wireless channel
  - Bio-medical signals
  - Speech signals

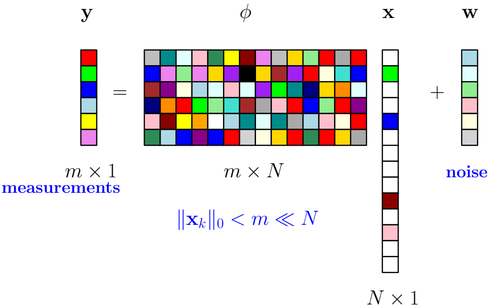


- Exploiting structures may improve recovery performance

Online Computation

- Complete input is not known in advance
- Input arrives incrementally, one piece at a time
- Advantages:
  - Improvements in terms of memory and computational speed
  - Cope with time-varying parameters

Sparse Signal Recovery Problem



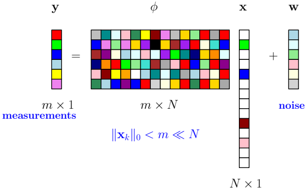
- **Goal**: Recover $\boldsymbol{x}$ from $\boldsymbol{y}$
- $m \ll N$: choose the sparsest solution from infinitely many solutions

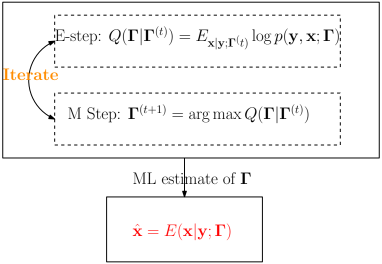
Sparse Bayesian Learning

- Impose a fictitious sparsity inducing prior on $\boldsymbol{x}$

$$\boldsymbol{x} \sim \mathcal{N}(0, \boldsymbol{\Gamma})$$
$$\boldsymbol{\Gamma} = \text{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$$

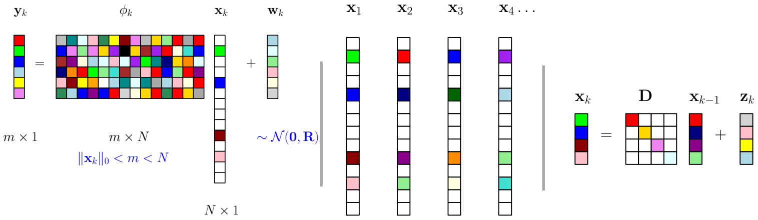
- Noise: $\boldsymbol{y}|\boldsymbol{x} \sim \mathcal{N}(\phi \boldsymbol{x}, \sigma^2 \boldsymbol{I})$





System Model

- Multiple measurement model



- Temporally correlated sparse vectors share same support
- **Goal**: Online recover the sparse vectors with a small time lag between measurements and estimation

SBL: Correlated MMV

- Impose a sparsity inducing prior on

$$\begin{aligned} \boldsymbol{x}_k &\sim \mathcal{N}(0, \boldsymbol{\Gamma}) \\ \boldsymbol{\Gamma} &= \operatorname{diag}\{\boldsymbol{\gamma}\} \end{aligned}$$

- **Problem**: Given K measurement vectors $\{\boldsymbol{y}_k\}_{k=1}^K$, estimate $\boldsymbol{\Gamma}$ and $\{\boldsymbol{x}_k\}_{k=1}^K$
  - If $\boldsymbol{\Gamma}$ is known, estimation becomes Kalman smoothing problem

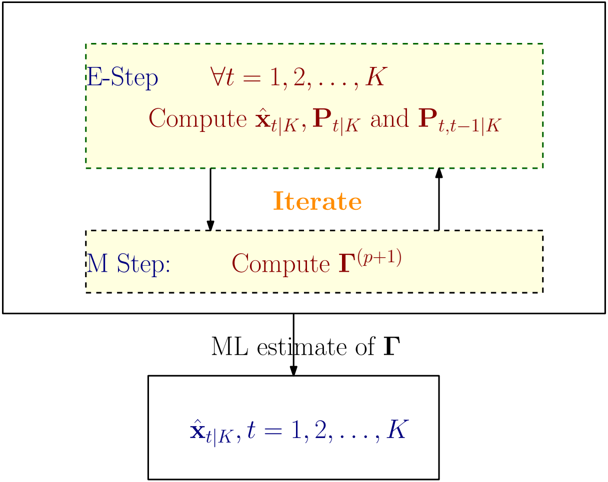
Offline SBL Approach

- Repeated applications of the E-step along with the M-step yield MLE of γ

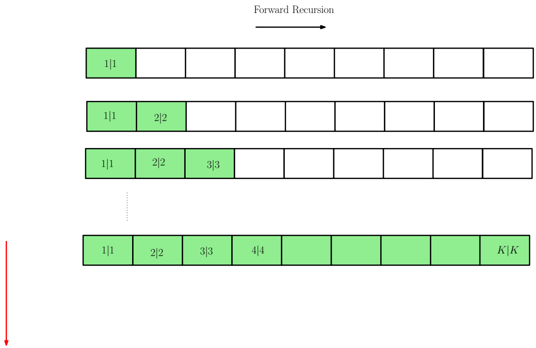
$$\begin{aligned}\textbf{E-step:}\mathcal{Q}(\gamma|\gamma^{(p)}) &= \mathbb{E}_{\boldsymbol{x}_{1:K}|\boldsymbol{y}_{1:K};\gamma^{(p)}}\left\{\log\left[p\left(\boldsymbol{y}_{1:K},\boldsymbol{x}_{1:K}\right)\right]\right\} \\ \textbf{M-step:}\gamma^{(p+1)} &= \arg\max_{\gamma}\mathcal{Q}(\gamma|\gamma^{(p)})\end{aligned}$$

- **M-step**:$\gamma^{(p+1)}$ to a closed form function of $\hat{\boldsymbol{x}}_{t|K}$, $\boldsymbol{P}_{t|K}$ and $\boldsymbol{P}_{t,t-1|K}$, $t = 1, 2, \ldots, K$
  - $\hat{\boldsymbol{x}}_{t|K} = \mathbb{E}\left\{\boldsymbol{x}_t|\boldsymbol{y}_{1:K}\right\}$
  - $\boldsymbol{P}_{t|K} = \mathbb{E}\left\{\boldsymbol{x}_t\boldsymbol{x}_t^{\mathsf{T}}|\boldsymbol{y}_{1:K}\right\}$
  - $\boldsymbol{P}_{t,t-1|K} = \mathbb{E}\left\{\boldsymbol{x}_t\boldsymbol{x}_{t-1}^{\mathsf{T}}|\boldsymbol{y}_{1:K}\right\}$
- **E-step**: Computes $\hat{\boldsymbol{x}}_{t|K}$, $\boldsymbol{P}_{t|K}$ and $\boldsymbol{P}_{t,t-1|K}$ using $\gamma^{(p)}$
  - Fixed interval Kalman smoothing

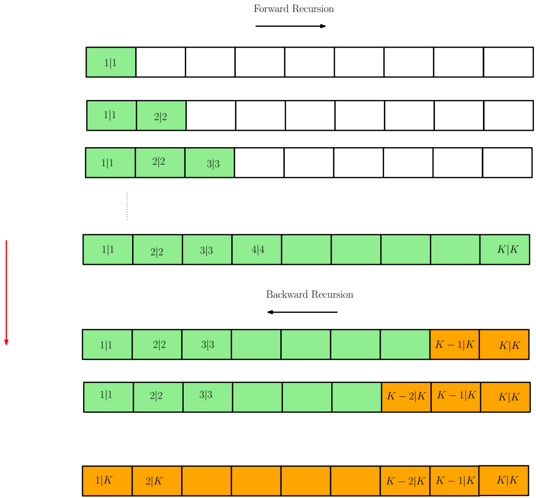
Offline SBL Algorithm



Fixed Interval Smoothing



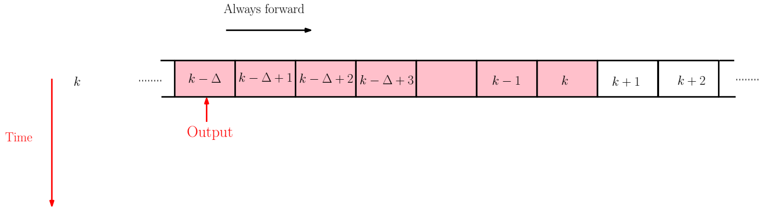
Fixed Interval Smoothing



Online Version

- Input arrives incrementally, one measurement at a time
- Time lag of Δ between the estimation and measurement
- Sequential EM algorithm implementation
- Approximations:
  - $\hat{\boldsymbol{x}}_{t|K} \approx \hat{\boldsymbol{x}}_{t|t+\Delta}$
  - $\boldsymbol{P}_{t|K} \approx \boldsymbol{P}_{t|t+\Delta}$
  - $\boldsymbol{P}_{t,t-1|K} \approx \boldsymbol{P}_{t,t-1|t+\Delta}$

Approach 1: Fixed Lag Smoothing



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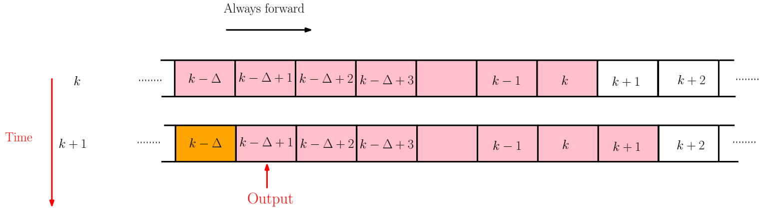

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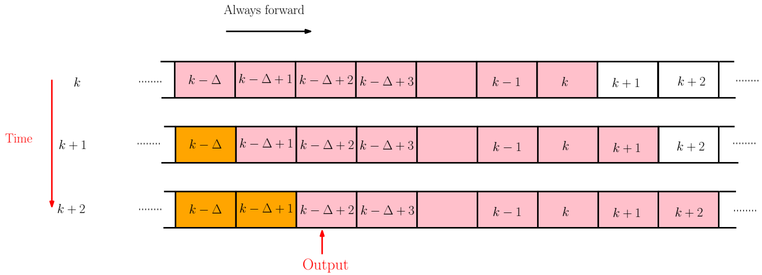


Approach 1: Fixed Lag Smoothing

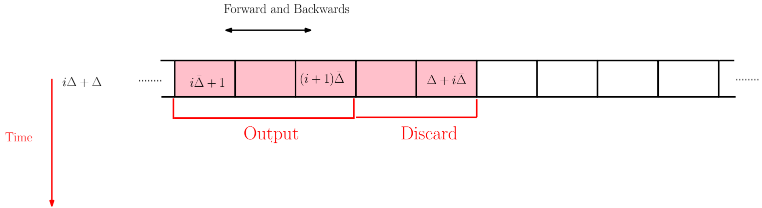
SBL with Fixed Lag Smoothing

- At time k
  1. Initialize γ (obtained at $k-1$)
  2. Run fixed lag algorithm to estimate $\hat{\boldsymbol{x}}_{t|k}$ for t from $k-\Delta$ to k
  3. Compute new γ
  4. If γ does not converge, go to step 2
  5. **Output**: $\hat{\boldsymbol{x}}_{k-\Delta|k}$

Approach 2: Sawtooth Lag Smoothing

- Compromise between fixed lag smoothing and fixed interval smoothing
- Waits for data of batch size $\bar{\Delta} = \Delta/2$
- Application of the fixed-interval smoothing scheme to overlapping sub-intervals
- Discard estimates with lag less that a threshold which are not sufficiently accurate

Approach 2: Sawtooth Lag Smoothing



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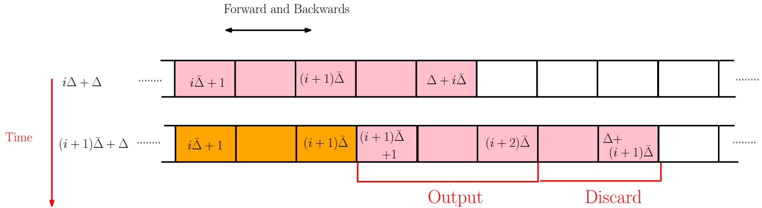
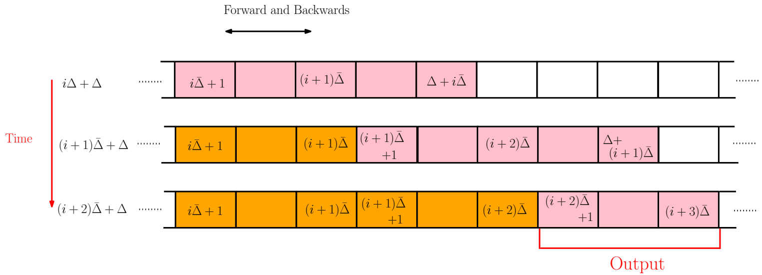


Approach 2: Sawtooth Lag Smoothing



SBL with Sawtooth Lag Smoothing

- At time $k = \Delta + i\bar{\Delta}$

  1. Initialize γ (obtained at $k-1$)
  2. Run Kalman filter algorithm to estimate $\hat{\boldsymbol{x}}_{t|t}$ for t from $k - \Delta + 1$ to k
  3. Implement the backward scheme to estimate $\hat{\boldsymbol{x}}_{t|k}$ for t from $k-1$ to $k - \Delta + 1$
  4. Discard estimates $\hat{\boldsymbol{x}}_{t|k}$ for t from $k - \Delta + \bar{\Delta}$ to k
  5. Compute new γ
  6. If γ does not converge, go to step 2
  7. **Output**: $\hat{\boldsymbol{x}}_{t|k}$ for t from $k - \Delta + 1$ to $k - \Delta + \bar{\Delta}$

Comparison Online Vs Offline

Smoothing	Computations	Memory	Average latency
Fixed Interval	$\mathcal{O}(KN^3)$	$\mathcal{O}\left(KN^2\right)$	K
Fixed Lag	$\mathcal{O}(K\Delta N^2 m)$	$\mathcal{O}(\Delta N^2)$	Δ
Sawtooth Lag	$\mathcal{O}(K\Delta N^3/\bar{\Delta})$	$\mathcal{O}\left(\Delta N^2\right)$	$\bar{\Delta}$

Recovery Performance

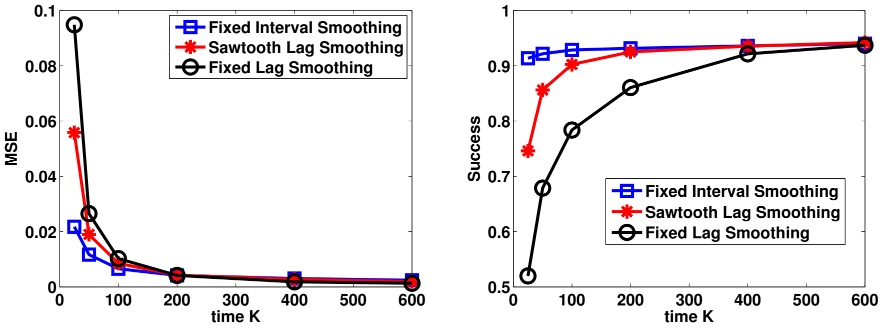


Figure: Setup: $N = 50, m = 20, s = 5, \Delta = 10$

Computational Cost

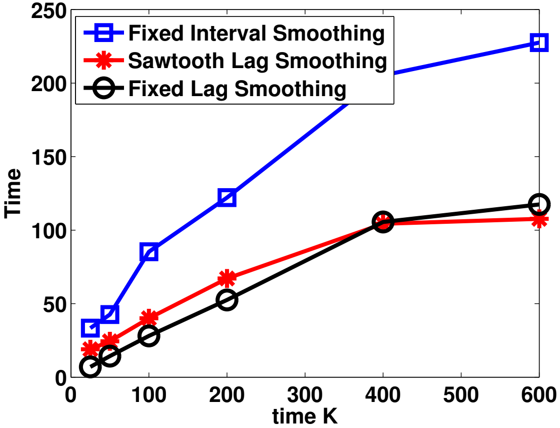


Figure: Setup: $N = 50, m = 20, s = 5, \Delta = 10$

Summary

- **Problem**: Recovery of correlated sparse vectors from noisy, underdertemined linear measurements
- **Online**: Small time lag is allowed between measurement and estimation
- **Approach**: Sparse Bayesian learning using sequential EM schemes
- **Proposals**: Fixed lag smoothing and Sawtooth lag smoothing

Fixed Interval Smoothing

Observations $\mathbf{y}_1, \ldots, \mathbf{y}_K$ →

Forward Recursions ($t = 1, 2, \ldots, K$):

$$\mathbf{G}_t = \mathbf{P}_{t|t-1}\phi_t' \left(\phi_t \mathbf{P}_{t|t-1}\phi_t' + \mathbf{R}\right)^{-1}$$

$$\mathbf{x}_{t|t} + \mathbf{D}\mathbf{G}_t \left(\mathbf{y}_t - \phi_t \mathbf{D}\mathbf{x}_{t-1|t-1}\right)$$

$$\mathbf{P}_{t|t} = (\mathbf{I} - \mathbf{G}_t\phi_t)\left(\mathbf{D}\mathbf{P}_{t-1|t-1}\mathbf{D} + \mathbf{\Gamma}(\mathbf{I} - \mathbf{D}^2)\right)$$

→

Backward Recursions ($t = K-1, K-2, \ldots, 1$):

$$\mathbf{J}_{t-1} = \mathbf{P}_{t-1|t-1}\mathbf{D}\mathbf{P}_{t|t-1}^{-1}$$

$$\hat{\mathbf{x}}_{t-1|K} = \mathbf{D}\hat{\mathbf{x}}_{t-1|t-2} + \mathbf{J}_{t-1}\left(\hat{\mathbf{x}}_{t|K} - \mathbf{D}\hat{\mathbf{x}}_{t|t-1}\right)$$

$$\mathbf{P}_{t-1|K} = \mathbf{P}_{t-1|t-1} + \mathbf{J}_{t-1}\left(\mathbf{P}_{t|K} - \mathbf{P}_{t|t-1}\right)\mathbf{J}_{t-1}'$$

$$\gamma^{(r+1)} = \frac{1}{k}\left(\boldsymbol{I} - \left[\boldsymbol{D}^{(r)}\right]^2\right)^{-1} \text{dia}\left\{\sum_{t=2}^{k} \boldsymbol{T}_{t|k} + \boldsymbol{P}_{1|k}\right\}.$$

Fixed Lag Filter

$$
\begin{aligned}
&\boldsymbol{J}_k = \phi_k^{\mathrm{T}} \left(\phi_k \boldsymbol{P}_{k|k-1} \phi_k^{\mathrm{T}} + \boldsymbol{R}\right)^{-1} \mathcal{O}(m^3) \\
&\boldsymbol{v}_k = \boldsymbol{y}_k - \phi_k \hat{\boldsymbol{x}}_{k|k-1} \mathcal{O}(N^2 m) \\
&\hat{\boldsymbol{x}}_{k|k-1} = \boldsymbol{D} \hat{\boldsymbol{x}}_{k-1|k-1} \mathcal{O}(Nm) \\
&\boldsymbol{P}_{k|k-1} = \boldsymbol{D} \boldsymbol{P}_{k-1|k-1} \boldsymbol{D} + \boldsymbol{\Gamma}(\boldsymbol{I} - \boldsymbol{D}^2) \mathcal{O}(N^2) \\
&\text{for } t = 0, 1, 2, \ldots, \Delta \text{ do} \\
&\qquad \boldsymbol{G}_k^{(t)} = \boldsymbol{P}_{k-t,k|k-1} \boldsymbol{J}_k \mathcal{O}(N^2 m) \\
&\qquad \hat{\boldsymbol{x}}_{k-t|k} = \hat{\boldsymbol{x}}_{k-t|k-1} + \boldsymbol{G}_k^{(t)} \boldsymbol{v}_k \mathcal{O}(Nm) \\
&\qquad \boldsymbol{P}_{k-t|k} = \boldsymbol{P}_{k-t|k-1} - \boldsymbol{G}_k^{(t)} \phi_k \boldsymbol{P}_{k,k-t|k-1} \mathcal{O}(N^2 m) \\
&\qquad \text{if } t \neq \Delta \\
&\qquad\qquad \boldsymbol{P}_{k-t,k-t-1|k} = \left(\boldsymbol{I} - \boldsymbol{G}_k^{(t)} \phi_k\right) \boldsymbol{P}_{k,k-t-1|k-1} \mathcal{O}(N^2 m) \\
&\qquad \boldsymbol{P}_{k+1,k-t|k} = \boldsymbol{D} \left(\boldsymbol{I} - \boldsymbol{G}_k^{(0)} \phi_k\right) \boldsymbol{P}_{k,k-t|k-1} \mathcal{O}(N^2 m) \\
&\qquad \text{end for}
\end{aligned}
$$

Sawtooth Lag Filter

Forward scheme:

for $t = k - \Delta + 2, k - \bar{\Delta} + 2, \ldots, k$

$$\boldsymbol{x}_{t|t-1} = \boldsymbol{D}\boldsymbol{x}_{t-1|t-1} \mathcal{O}(N)$$

$$\boldsymbol{P}_{t|t-1} = \boldsymbol{D}\boldsymbol{P}_{t-1|t-1}\boldsymbol{D} + \boldsymbol{\Gamma}(\boldsymbol{I} - \boldsymbol{D}^2) \mathcal{O}(N^2)$$

$$\boldsymbol{G}_t = \boldsymbol{P}_{t|t-1}\phi_t^{\mathrm{T}} \left(\phi_t \boldsymbol{P}_{t|t-1}\phi_t^{\mathrm{T}} + \boldsymbol{R}\right)^{-1} \mathcal{O}(N^2 m + m^3)$$

$$\hat{\boldsymbol{x}}_{t|t} = \boldsymbol{x}_{t|t-1} + \boldsymbol{D}\boldsymbol{G}_t \left(\boldsymbol{y}_t - \phi_t \boldsymbol{x}_{t|t-1}\right) \mathcal{O}(Nm)$$

$$\boldsymbol{P}_{t|t} = (\boldsymbol{I} - \boldsymbol{G}_t\phi_t)\,\boldsymbol{P}_{t|t-1} \mathcal{O}(N^2 m)$$

end for

$$\boldsymbol{P}_{k,k-1|k} = (\boldsymbol{I} - \boldsymbol{G}_k\phi_k)\,\boldsymbol{D}\boldsymbol{P}_{k-1|k-1} \mathcal{O}(N^2 m)$$

Backward scheme:

for $t = k, k-1, \ldots k - \Delta + 2$

$$\boldsymbol{J}_{t-1} = \boldsymbol{P}_{t-1|t-1}\boldsymbol{D}\boldsymbol{P}_{t|t-1}^{-1} \mathcal{O}(N^3)$$

$$\hat{\boldsymbol{x}}_{t-1|k} = \boldsymbol{D}\hat{\boldsymbol{x}}_{t-1|t-2} + \boldsymbol{J}_{t-1}\left(\hat{\boldsymbol{x}}_{t|k} - \hat{\boldsymbol{x}}_{t|t-1}\right) \mathcal{O}(N^2)$$

$$\boldsymbol{P}_{t-1|k} = \boldsymbol{P}_{t-1|t-1} + \boldsymbol{J}_{t-1}\left(\boldsymbol{P}_{t|k} - \boldsymbol{P}_{t|t-1}\right)\boldsymbol{J}_{t-1}^{\mathrm{T}} \mathcal{O}(N^3)$$

if $t \neq k$

$$\boldsymbol{P}_{t,t-1|k} = \boldsymbol{P}_{t|t}\boldsymbol{J}_{t-1}^{\mathrm{T}} + \boldsymbol{J}_t\left(\boldsymbol{P}_{t+1,t|k} - \boldsymbol{D}\boldsymbol{P}_{t|t}\right)\boldsymbol{J}_{t-1}^{\mathrm{T}} \mathcal{O}(N^3)$$

end for