Physical Layer Binary Consensus Protocols: A Study

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Outline

- Introduction to Consensus
- Problem Setup
- Physical Layer Binary Consensus Protocols
    - LMMSE-based scheme
    - Co-phased combining based scheme
- Performance Analysis
- Simulation Results

Consensus Problems

- A set of nodes with arbitrary initial data values agree upon a common value
  - Examples: min., max., average, majority value
- Nodes repeatedly exchange msgs & update their values
- Network layer consensus
  - Reliable packet exchanges in the local neighborhood
- Physical layer consensus
  - Data exchanges with all other nodes over noisy wireless links
  - No overhead of control information

Literature Survey: Network Layer Consensus

- Distributed averaging
- [Tsitsiklis 1984] Distributed computing
- [Boyd et al. 2005] Gossip algorithms
- [Benezit et al. 2011] Voting problem as interval consensus

Literature Survey: Physical Layer Consensus

- Distributed detection
- [Oltafi 2006, Chan 2010, Wang 2012] Consensus on test statistic and treat as distributed hypothesis testing
- [Mostofi 2007, 2008, 2010] Exchange hard decisions by broadcasting bits and attain majority consensus
- Our focus
  - Physical layer binary consensus by exchanging hard decisions
  - Broadcast-based bit-exchange
  - Two bit-update schemes: (a) LMMSE-based scheme and (b) Co-phased combining scheme

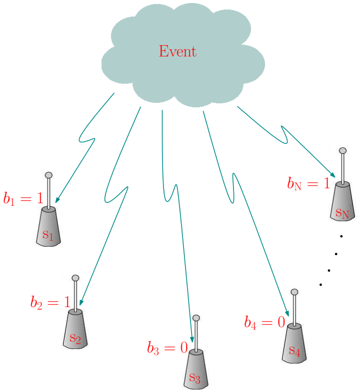
Contributions

- Performance analysis:
  - Probability of correct majority bit detection for co-phased combining scheme
  - Average hitting time
  - Average consensus duration
- Analysis captures the effect of channel estimation errors, fading, and noise on consensus performance

Main Message

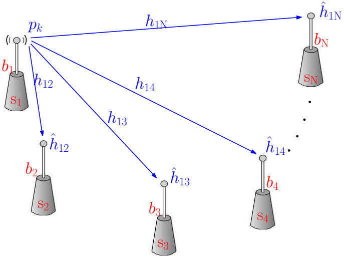
Simple co-phased combining scheme has advantage over conventional LMMSE-based scheme at low to moderate pilot SNRs.

Problem Setup



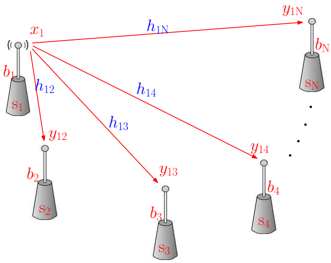
- Nodes $[s_1, s_2, \ldots, s_N]$ have initial values $[b_1, b_2, \ldots, b_N]$
- Goal: To achieve majority consensus

Broadcast-based Data Exchange: Pilot Phase



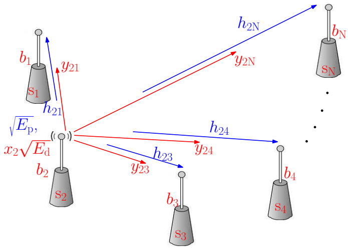
- Node broadcasts a known pilot symbol p_k
- All other nodes estimate the corresponding channel

Broadcast-based Data Exchange: Data Phase



- Node broadcasts a BPSK symbol x_i corresp. to data bit b_i
- At node s_j, $y_{1j} = h_{1j}x_1 + w_j$, where $w_j \sim \mathcal{CN}(0, \sigma^2)$

Broadcast-based Data Exchange



- A bit exchange cycle: nodes broadcast a pilot symbol followed by a data bit, in a round-robin manner
- At the end of a cycle, node s_j will have $\{y_{ij}\}_{i=1,\dots,N, i\neq j}$
- Bit-update based on these samples

Multiple Cycles of Bit Exchanges and Update

- Bit exchanges happen over noisy fading channels
- Multiple cycles are required to achieve consensus
- Define network state $[b_1(t)\ b_2(t)\ \ldots\ b_N(t)]$ collection of decision bits at the N nodes
- After every update cycle, network will be in one of the $M = 2^N$ states
  - The all-zero and all-one states are consensus states
- Current network state depends on previous network state, current channel states, and current receiver noise: Markovian evolution

Network State Evolution as a Markov chain

- State distribution vector: $\pi(t) = \mathbf{P}(t)\pi(t-1)$; $\mathbf{P}(t)$ is the one-step transition probability matrix (tpm)
- Leads to: $\pi(t) = \mathbf{P}(t)\mathbf{P}(t-1)\ldots\mathbf{P}(1)\pi(0)$, i.e., a time *inhomogeneous* Markov chain
- In such scenarios, the average tpm is considered, $\bar{\pi}(t) = (\bar{\mathbf{P}})^t\pi(0)$
- The average tpm is *irreducible*. Thus, the stationary state distribution is independent of the initial state
  - Memoryless consensus: the final consensus state is independent of the initial state of the system
  - This is bad news!

Good News: Transient Period of the Markov Chain

- During the initial transient period: the network reaches accurate consensus with high probability
- Largest eigen value of the tpm is 1
- Second largest eigen value of the tpm: the closer it is to 1, the longer the transient period
- Need a way to decide when to stop the consensus procedure
- Average hitting time and average consensus duration

Ex: Avg. TPM ($\bar{\mathbf{P}}$)

- $N = 3$
- l^{th} column represents transition probs. from state $\phi^{(l)}$ to all other states

0.9884	0.2490	0.2490	0.0010	0.2490	0.0010	0.0010	0.0000
0.0039	0.0010	0.2490	0.0010	0.2490	0.0010	0.2490	0.0000
0.0039	0.2490	0.0010	0.0010	0.2490	0.2490	0.0010	0.0000
0.0000	0.0010	0.0010	0.0010	0.2490	0.2490	0.2490	0.0039
0.0039	0.2490	0.2490	0.2490	0.0010	0.0010	0.0010	0.0000
0.0000	0.0010	0.2490	0.2490	0.0010	0.0010	0.2490	0.0039
0.0000	0.2490	0.0010	0.2490	0.0010	0.2490	0.0010	0.0039
0.0000	0.0010	0.0010	0.2490	0.0010	0.2490	0.2490	0.9884

$\bar{\mathbf{P}}^{20}$

0.9034	0.6185	0.6185	0.3348	0.6185	0.3348	0.3348	0.0523
0.0096	0.0085	0.0085	0.0070	0.0085	0.0070	0.0070	0.0052
0.0096	0.0085	0.0085	0.0070	0.0085	0.0070	0.0070	0.0052
0.0052	0.0070	0.0070	0.0085	0.0070	0.0085	0.0085	0.0096
0.0096	0.0085	0.0085	0.0070	0.0085	0.0070	0.0070	0.0052
0.0052	0.0070	0.0070	0.0085	0.0070	0.0085	0.0085	0.0096
0.0052	0.0070	0.0070	0.0085	0.0070	0.0085	0.0085	0.0096
0.0523	0.3348	0.3348	0.6185	0.3348	0.6185	0.6185	0.9034

$\bar{\mathbf{P}}^{400}$

0.4997	0.4851	0.4851	0.4705	0.4851	0.4705	0.4705	0.4559
0.0075	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0073
0.0075	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0073
0.0073	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0075
0.0075	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0073
0.0073	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0075
0.0073	0.0074	0.0074	0.0074	0.0074	0.0074	0.0074	0.0075
0.4559	0.4705	0.4705	0.4851	0.4705	0.4851	0.4851	0.4997

Received Samples at Node s_j

- Node s_i broadcasts a pilot symbol of power E_p

$$y_{ij}^{(p)} = h_{ij}\sqrt{E_p} + w_{ij}^{(p)}$$

$$\hat{h}_{ij} = \frac{y_{ij}^{(p)}}{\sqrt{E_p}} = h_{ij} + \frac{w_{ij}^{(p)}}{\sqrt{E_p}}.$$

- Node s_i broadcasts data symbol $x_i\sqrt{E_d}$

$$y_{ij}^{(d)} = h_{ij}x_i\sqrt{E_d} + w_{ij}^{(d)}, \text{ where } x_i = \begin{cases} +1 & b_i(t-1) = 1 \\ -1 & b_i(t-1) = 0 \end{cases}$$

- Processed samples at node s_j

$$r_{ij} = \mathrm{Re}\{y_{ij}^{(d)} e^{-j\hat{\theta}_{ij}}\} \triangleq |h_{ij}| \cos\tilde{\theta}_{ij} x_i \sqrt{E_d} + v_{ij}$$
$$\text{for } i \in \{1, 2, \ldots, N\},\ i \neq j,\ v_{ij} \sim \mathcal{CN}(0, \sigma_w^2)$$

Majority Bit Detection

- Since BPSK is employed, the sum of votes $\Delta_j \triangleq \sum_{\substack{i=1 \\ i \neq j}}^{N} x_i$ is a test statistic for detecting majority bit[1]
- Sum of votes estimate depends on $\{r_{1j}, \ldots, r_{ij}, \ldots, r_{Nj}\}_{i \neq j}$
- Majority bit detection rule

$$g(\hat{\Delta}_j) = \begin{cases} 1 & \hat{\Delta}_j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

[1] For simplicity, the self-bit, i.e., the sensor's own observation, is ignored here.

LMMSE-based Scheme

- Sum of votes estimate, $\hat{\Delta}_j^{(wc)} = \boldsymbol{\alpha}_j^T \mathbf{r}_j$
- LMMSE estimate, $\boldsymbol{\alpha}_j^* = \arg\min_{\boldsymbol{\alpha}_j} \mathbb{E}[(\hat{\Delta}_j^{(wc)} - \Delta_j)^2]$
- The MMSE solution $\alpha_{ij}^* = \frac{|\hat{h}_{ij}|}{|\hat{h}_{ij}|^2 + \sigma_w^2/2}$
- $\hat{\Delta}_j^{(wc)} = \sum_{i=1,\ i \neq j}^{N} \left[\alpha_{ij}^* |h_{ij}| \cos \tilde{\theta}_{ij} x_i \sqrt{E_d} + \alpha_{ij}^* v_{ij} \right]$

Co-phased Combining Scheme

- $\hat{\Delta}_j^{(cc)} = \sum_{i=1,\ i \neq j}^{N} r_{ij} = \sum_{i=1,\ i \neq j}^{N} \left[|h_{ij}| \cos \tilde{\theta}_{ij} x_i \sqrt{E_d} + v_{ij} \right]$
- Co-phased combining is a special case of LMMSE-based scheme
- $\hat{\Delta}_j = h\sqrt{E_d} + v$, where
  - $h \triangleq \sum_{i=1,\ i \neq j}^{N} \beta_{ij} |h_{ij}| \cos \tilde{\theta}_{ij} x_i$
  - $v \sim \mathcal{N}\left(0, \sigma_v^2\right)$ with $\sigma_v^2 = \sum_{i=1,\ i \neq j}^{N} (\beta_{ij})^2 \sigma_w^2 / 2$
  - $\beta_{ij} = \alpha_{ij}^*$ for the LMMSE-based scheme, and $\beta_{ij} = 1$ for the co-phased combining scheme.

Probability of Detecting the Majority Bit

- Probability of detecting bit 1 conditioned on effective channel H, $p_j \triangleq \Pr\{\hat{\Delta}_j \geq 0 | H = h\}$
- $\bar{p}_j = \int_{-\infty}^{\infty} \mathcal{Q}\left(\frac{-h\sqrt{E_d}}{\sigma_v}\right) f_H(h)\,\mathrm{d}h$
- $f_H(h)$ is tractable in case of co-phased combining scheme
- $h = h_p - h_n$, where h_p and h_n are effective channels corresponding to sensors transmitting a $+1$ and -1, respectively
- H_p is approximated by a Nakagami r.v. with parameters, $m_1 = (\mathbb{E}[H_p^2])^2/\mathrm{Var}[H_p^2]$ and $\Omega_1 = \mathbb{E}[H_p^2]$

Lemma

For a given pilot SNR, $\gamma_p \triangleq E_p/\sigma_w^2$ *and with the second moment of i.i.d. Rayleigh r.v.s* H_{ij}, $\mathbb{E}[H_{ij}^2] = \sigma^2$, *the second moment of r.v.* H_p *and variance of r.v.* H_p^2 *is given by*

$$\mathbb{E}[H_p^2] = \frac{K\sigma^2\left(2 + (4 + (K-1)\pi)\gamma_p\sigma^2\right)}{1 + \gamma_p\sigma^2}$$

$$\begin{aligned}\mathbf{Var}[H_p^2] &= K\mathbb{E}[|H_{ij}|^4\cos^4\tilde{\Theta}_{ij}] + 3K(K-1)(\mathbb{E}[|H_{ij}|^2\cos^2\tilde{\Theta}_{ij}])^2\\ &+ K(K-1)(K-2)(K-3)(\mathbb{E}[|H_{ij}|\cos\tilde{\Theta}_{ij}])^4\\ &+ 6K(K-1)(K-2)(\mathbb{E}[|H_{ij}|\cos\tilde{\Theta}_{ij}])^2\mathbb{E}[|H_{ij}|^2\cos^2\tilde{\Theta}_{ij}]\\ &+ 4K(K-1)\mathbb{E}[|H_{ij}|^3\cos^3\tilde{\Theta}_{ij}]\mathbb{E}[|H_{ij}|\cos\tilde{\Theta}_{ij}] - (\mathbb{E}[H_p^2])^2\end{aligned}$$

Lemma

The pdf of the effective channel, H, which is the difference of two Nakagami r.v.s H_p and H_n with parameters m_1, Ω_1 and m_2, Ω_2, is given by

$$f_H(h) = \frac{2\left(\frac{m_1}{\Omega_1}\right)^{m_1}\left(\frac{m_2}{\Omega_2}\right)^{m_2} \mathrm{e}^{-\frac{h^2 m_1 m_2}{m}}}{\Gamma(m_1)\Gamma(m_2)\left(\frac{m}{\Omega}\right)^{m_1+m_2-\frac{1}{2}}} \sum_{k=0}^{2m_1-1}\sum_{l=0}^{2m_2-1}\binom{2m_1-1}{k}\binom{2m_2-1}{l}$$
$$\left(\frac{m_2\Omega_1 h}{\sqrt{m\Omega}}\right)^{2m_1-1-k}\left(\frac{-m_1\Omega_2 h}{\sqrt{m\Omega}}\right)^{2m_2-1-l}\Gamma\left(\frac{k+l+1}{2}, \frac{(m_1\Omega_2 h)^2}{m\Omega}\right)$$

where $m \triangleq m_1\Omega_2 + m_2\Omega_1$, $\Omega \triangleq \Omega_1\Omega_2$ and $\Gamma(.,.)$ is the upper incomplete Gamma function. For $h < 0$, $f_H(h)$ can be evaluated by swapping the parameters m_1, Ω_1 with m_2, Ω_2, respectively.

Average Transition Probability Matrix

- Suppose the network is in a state $\phi^{(l)}$ at time $t-1$ and $\phi^{(k)} \triangleq [b_1^{(k)} \;\; b_2^{(k)} \;\; \ldots \;\; b_N^{(k)}]$ at time t
- $\bar{P}_{kl} = \prod_{j=1}^{N} \left[b_j^{(k)} \bar{p}_j^{(l)} + (1 - b_j^{(k)})(1 - \bar{p}_j^{(l)}) \right]$, where $\bar{p}_j^{(l)}$ is $\bar{p}_j$ conditioned on $\phi^{(l)}$
- Past result: Approximation to second eigen value (λ_2) is $1 - 2\bar{p}_j^{(1)}$, where $\bar{p}_j^{(1)}$ is $\bar{p}_j$ conditioned on the all-zero state
- When $N = 2$ or 3 sensors, $\lambda_2 = 1 - 2\bar{p}_j^{(1)}$

Average Hitting Time

- Average hitting time: average number of cycles required to reach consensus state for the first time
- $f_{ij}^{(n)}$ prob. of starting from state i and hitting state j in n cycles
- $[f_{ij}^{(n)}]_{i=1,\ldots,N} = \mathbf{Q}^{n-1}[p_{ij}]_{i=1,\ldots,N}$
- $\mathbf{Q}$ = matrix formed by removing j^{th} column of $\bar{\mathbf{P}}^T$
- $[p_{ij}]_{i=1,\ldots,N} = j^{th}$ column of $\bar{\mathbf{P}}^T$
- $\tau_h = \sum_{n=1}^{\infty} n f_{ij}^{(n)}$

Average Consensus Duration

- Average consensus duration: average number of cycles for which the network stays in consensus state

$$\tau_c = \sum_{n=1}^{\infty} n\left(\bar{P}_c\right)^n (1 - \bar{P}_c) = \frac{\bar{P}_c}{1 - \bar{P}_c}$$

- $\bar{P}_c =$ average probability of remaining in consensus after the next cycle, once the network is already in consensus

Simulation Setup

- Number of nodes, $N = 8$
- Channel coefficients, $h_{ij} \sim \mathcal{CN}(0, 1)$
- Receiver noise, $w_j \sim \mathcal{CN}(0, 1)$
- Averaged over 20000 instantiations

Second Eigen Value Vs. Data SNR

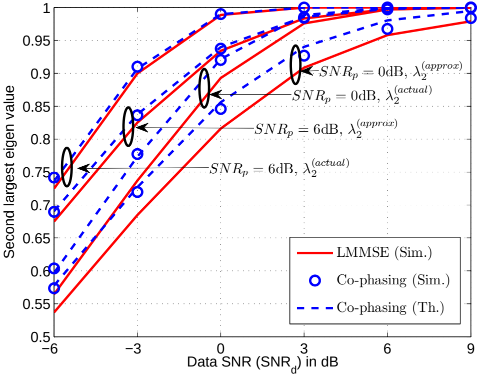
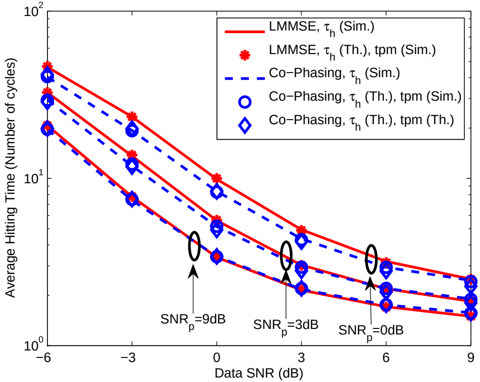


Figure: Second largest eigen value vs. Data SNR (denoted by $\mathrm{SNR_d}$) for different pilot SNRs (denoted by $\mathrm{SNR_p}$).

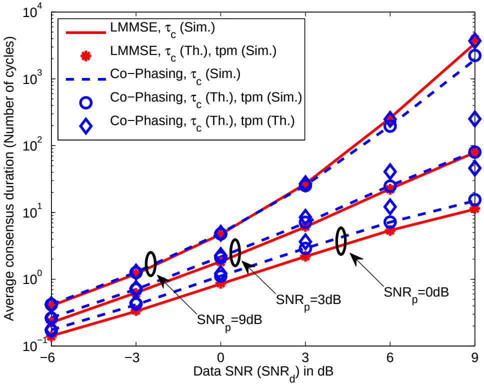
- λ_2 approximation is a lower bound on actual value

Average Hitting Time Vs. Data SNR



- Close agreement between theoretical and simulated curves

Average Consensus Duration Vs. Data SNR



- At low to intermediate pilot SNRs, co-phased combining scheme performs better

Probability of accurate consensus vs. number of consensus cycles

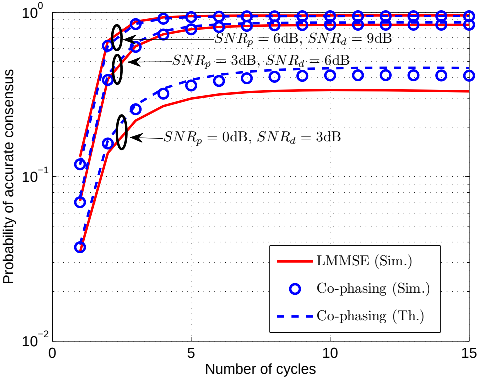


Figure: Probability of accurate consensus vs. number of consensus cycles, starting from an initial state of '00011111'.

Summary

- Compared LMMSE-based scheme and co-phased combining scheme in terms of average hitting time and average consensus duration
- Analyzed the average prob. of incorrect majority bit detection performance for co-phased combining scheme
- At low to intermediate pilot SNRs, co-phasing-based consensus outperforms the LMMSE scheme

Thank You