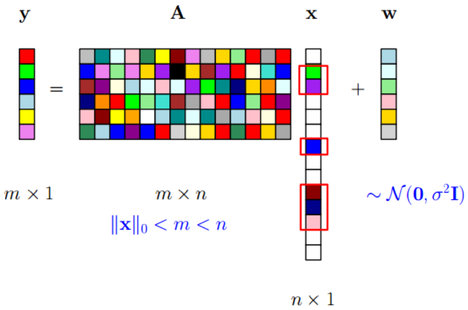
Block Sparse Signal Recovery using Tikhonov Regularization

Sai Subramanyam Thoota
Signal Processing for Communications Lab
Department of ECE, IISc

October 17, 2015

Block Sparse Signal Recovery Problem



- Goal: To recover the block sparse signal x from y
- Block boundaries and sizes are unknown

Current Approaches

- Sparse Bayesian Learning
  - Imposing a sparsity inducing prior on the vector *x*
    $\mathrm{x} \sim \mathcal{N}(0, \Gamma)$
  - Hyperparameters γ estimated using evidence maximization or type-II ML [1], [2]
  - Posterior density of the weights [2] is given by

$$\mathrm{p}(\mathrm{x}|\mathrm{y}; \gamma, \sigma^2) = \mathcal{N}(\mu, \Sigma_\mathrm{x}) \tag{1}$$

    where
    $\Sigma_\mathrm{x} = (\sigma^{-2}\Phi^\mathrm{T}\Phi + \Gamma^{-1})^{-1}$
    $\mu = \sigma^{-2}\Sigma_\mathrm{x}\Phi^\mathrm{T}\mathrm{y}$
  - SBL Objective function: To maximize $\mathrm{p}(\mathrm{y}; \gamma, \sigma^2)$

$$\mathrm{L} = \log|\Sigma_\mathrm{t}| + \mathrm{y}^\mathrm{T}\Sigma_\mathrm{t}^{-1}\mathrm{y} \tag{2}$$

    where $\Sigma_\mathrm{t} = (\sigma^2\mathrm{I} + \Phi\Gamma\Phi^\mathrm{T})$

Current Approaches

- Iterative Reweighted ℓ_1 Minimization [4]

$$\mathrm{x}^{(\mathrm{k}+1)} = \arg\min_{\mathrm{x}} \|\mathrm{y} - \Phi \mathrm{x}\|^2 + \lambda \sum_{\mathrm{i}} \mathrm{w}_{\mathrm{i}}^{(\mathrm{k})} |\mathrm{x}_{\mathrm{i}}| \qquad (3)$$

- $\mathrm{x}_{\mathrm{SBL}}$ satisfies the below equation [3]

$$\mathrm{x}_{\mathrm{SBL}} = \arg\min_{\mathrm{x}} \|\mathrm{y} - \Phi \mathrm{x}\|^2 + \lambda \mathrm{g}_{\mathrm{SBL}}(\mathrm{x}) \qquad (4)$$

where $\mathrm{g}_{\mathrm{SBL}}(\mathrm{x}) = \min_{\gamma \geq 0} \mathrm{x}^{\mathrm{T}} \Gamma^{-1} \mathrm{x} + \log|\alpha \mathrm{I} + \Phi \Gamma \Phi^{\mathrm{T}}|$

- $\mathrm{g}_{\mathrm{SBL}}(\mathrm{x})$ is a non-decreasing, concave function of $|\mathrm{x}|$ and can be optimized using a reweighted ℓ_1 algorithm

$$\mathrm{g}_{\mathrm{SBL}}(\mathrm{x}) = \min_{\gamma, \mathrm{z} \geq 0} \mathrm{x}^{\mathrm{T}} \Gamma^{-1} \mathrm{x} + \mathrm{z}^{\mathrm{T}} \gamma - \mathrm{h}^*(\mathrm{z}) \qquad (5)$$

where $\mathrm{h}^*(\mathrm{z})$ is the concave conjugate of
$\mathrm{h}(\gamma) = \log|\alpha \mathrm{I} + \Phi \Gamma \Phi^{\mathrm{T}}|$ given by

$$\mathrm{h}^*(\mathrm{z}) = \min_{\gamma \geq 0} \mathrm{z}^{\mathrm{T}} \gamma - \log|\alpha \mathrm{I} + \Phi \Gamma \Phi^{\mathrm{T}}| \qquad (6)$$

Proposed Approaches

- Solution 1: Tikhonov Regularizer term along with the SBL regularizer

$$\mathrm{g_{SBL}(x) = min_{\gamma,z\geq 0} x^T\Gamma^{-1}x + z^T\gamma - h^*(z) + \|L\gamma\|_2^2} \qquad (7)$$

where

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- Numerical Method approach since there is no analytical solution for the problem. Newton's Method of solving simultaneous linear equations
- Solution 2: ℓ_1 regularizer instead of ℓ_2 regularizer to impose block sparsity constraint on the vector $\mathrm{L}\gamma$ (Solution was computationally complex and difficult to solve)

Proposed Approaches

- Conventional Expectation Maximization (EM) based Approach (for SBL)
  - Objective Function:

$$E_{x|y,\gamma^{(k)},\sigma^2}[\log(p(y,x;\gamma,\sigma^2)] = E_{x|y,\gamma^{(k)},\sigma^2}[\log(p(x;\gamma))] \quad (8)$$

  - E Step: Treat x as hidden variables

$$E_{x|y,\gamma^{(k)},\sigma^2}[x_i^2] = (\Sigma_x)_{i,i} + \mu_i^2 \quad (9)$$

  - M Step

$$\gamma_i^{k+1} = \text{argmax}_{\gamma_i \geq 0} E_{x|y,\gamma^{(k)},\sigma^2}[x_i^2] \quad (10)$$

    where
    $\Sigma_x = (\sigma^{-2}\Phi^T\Phi + \Gamma^{-1})^{-1}$
    $\mu = \sigma^{-2}\Sigma_x\Phi^T y$

- Prior Model:

$$\gamma = [\epsilon_1, \epsilon_1 + \epsilon_2, ..., \sum_i \epsilon_i]^T \quad (11)$$

  Constraint: $\sum_i \epsilon_i \geq 0$

EM based Approach

- Solution of the prior model shown above is same as that of the SBL
- ℓ_1 Regularizer

$$\mathrm{argmin}_{\epsilon} \sum_{i=1}^{N} \frac{b_i}{\sum_{j=1}^{i} \epsilon_j} + \log\Big(\sum_{j=1}^{i} \epsilon_j\Big) + \lambda \|\epsilon\|_1 \qquad (12)$$

where $b_i = (\Sigma_x)_{i,i}$

- Constrained optimization approach being derived to solve this problem. Solving N simultaneous equations with the constraint that $\sum_{j=1}^{i} \geq 0$ for all $i = 1, 2, ..., N$

References

1 R. Neal, Bayesian Learning for Neural Networks. New York: Springer-Verlag, 1996.

2 M. Tipping, "Sparse Bayesian Learning and the relevance vector machine." J. Machine Learning Research, Vol. 1, pp. 211-244, 2001.

3 D. Wipf and S. Nagarajan, "A new view of automatic relevance determination," Advances in Neural Information Processing Systems 20, 2008.

4 David Wipf and Srikantan Nagarajan, "Iterative Reweighted ℓ_1 and ℓ_2 Methods for Finding Sparse Solutions." IEEE Journal of Selected Topics in Signal Processing, Vol. 4, No. 2, April 2010.