E2 212: Homework - 5

1 Topics

- Linear least-squares estimation, properties of the solution (maximum-likelihood, CRLB)
- The full-rank LS problem: Normal equations
- The rank-deficient LS problem: solution via SVD, pseudo-inverses
- Principal component analysis, trade-off between bias and variance
- QR decomposition - Gram-Schmidt, Householder, Givens Transforms

2 Problems

1. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\mathbf{x} = (\mathbf{x}_i) \in \mathbb{R}^n$ and $\mathbf{b} = (\mathbf{b}_i) \in \mathbb{R}^n$. Show that

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \sum_{k=1}^{m} (\sum_{j=1}^{n} a_{kj}\mathbf{x}_j - \mathbf{b}_k)^2.$$

Show that the equations $\partial f/\partial \mathbf{x}_i = 0$, $i = 1, \ldots, n$, where $f(\mathbf{x}) \triangleq \|A\mathbf{x} - \mathbf{b}\|^2$, are equivalent to the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$.

2. Assume that the measurement vector $\mathbf{y}$ is obtained from $\mathbf{b}$ by,

$$\mathbf{y} = W\mathbf{b} + \mathbf{n},$$

where $\mathbb{E}(\mathbf{b}^H\mathbf{b}) = R$, $\mathbb{E}(\mathbf{n}\mathbf{n}^H) = Q$ and $\mathbb{E}(\mathbf{b}\mathbf{n}^H) = S$, and W is a known matrix of appropriate dimensions. Show that the minimum mean square estimate of $\mathbf{b}$ based on $\mathbf{y}$ is

$$\hat{\mathbf{b}} = (RW^H + S)(WRW^H + WS + S^H W^H + Q)^{-1}\mathbf{y}.$$

3. The regularized weighted least squares problem seeks a vector $\hat{\mathbf{w}}$ that solves the following:

$$\min_{\mathbf{w}} \left[\mathbf{w}^H \Pi \mathbf{w} + (\mathbf{y} - H\mathbf{w})^H W(\mathbf{y} - H\mathbf{w})\right],$$

where $\Pi > 0$, $W \geq 0$. Show that the solution to the above problem is

$$\hat{\mathbf{w}} = (\Pi + H^H W H)^{-1} H^H W\mathbf{y}.$$

4. Given $A \in \mathbb{R}^{m\times n}$, and a set of vectors $\mathbf{x}_i \in \mathbb{R}^m$, $\mathbf{y}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, k$,
   (a) Find a set of coefficients a_i such that $\|A - \sum_{i=1}^{k} a_i\mathbf{x}_i\mathbf{y}_i^T\|_F^2$ is minimized.
   (b) What are the set of $\mathbf{x}_i$, $\mathbf{y}_i$ that minimize the minimum in (a) for $k < \min(m, n)$?

5. Given the system of equations $A\mathbf{x} = \mathbf{b}$, A being a tall matrix, what is P such that $\left[\min_{\mathbf{x}} \|A\mathbf{x} - P\mathbf{b}\|_2^2\right]$ is minimized?

6. A satellite is put into orbit about the earth. Its initial state vector (whose components are positions and velocity coordinates) is not known precisely and it is desired to estimate it from later measurements of the satellite position. The equations of motion are:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k),$$

where $\mathbf{x}(k)$ is the n dimensional state vector and Φ is an $n \times n$ matrix. It is assumed that $\mathbb{E}(\mathbf{x}(0)) = \mathbf{0}$, $\mathbb{E}(\mathbf{x}(0)\mathbf{x}(0)^H) = P$. The measurements are of the form

$$\mathbf{v}(k) = M\mathbf{x}(k) + \mathbf{n}(k),$$

where M is an $m \times n$ matrix ($m < n$) and $\mathbb{E}(\mathbf{n}(k)\mathbf{n}(j)^H) = Q\delta_{jk}$. Develop the recursive equations for the minimum variance linear estimate of $\mathbf{x}(0)$ given the observations.

7. Let $\hat{\mathbf{w}}$ be the solution to

$$\min_{\mathbf{w}} \|\mathbf{y} - H\mathbf{w}\|_2^2$$

and let $\hat{\mathbf{y}} \triangleq H\hat{\mathbf{w}}$. Show that

   (a) The fundamental orthogonality principle holds, i.e., $\hat{\mathbf{w}}$ is a solution if, and only if, the residual $\tilde{\mathbf{y}} \triangleq \mathbf{y} - H\hat{\mathbf{w}}$ is orthogonal to $\mathcal{R}(H)$.

   (b) The norms of $\{\mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}\}$ satisfy the relation $\|\mathbf{y}\|^2 = \|\hat{\mathbf{y}}\|^2 + \|\tilde{\mathbf{y}}\|^2$.

8. Show that solution of rank deficient LS problem is not unique. Also show that set of all minimizers is convex, i.e. if $\mathbf{x}_1$ and $\mathbf{x}_2$ minimizes $\|A\mathbf{x} - \mathbf{b}\|$, then so does $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for $\lambda \in [0, 1]$.

9. Define $B(\lambda) \in \mathbb{R}^{n\times m}$ by $B(\lambda) = (A^TA + \lambda I)^{-1}A^T$, $\lambda > 0$. Show

$$\|B(\lambda) - A^+\|_2 = \frac{\lambda}{\sigma_r(A)[(\sigma_r(A))^2 + \lambda]}$$

10. Let $\mathbf{x}$ and $\mathbf{y}$ be two nonzero vectors in $\mathbb{R}^n$. Give an algorithm for determining Householder matrix P such that $P\mathbf{x}$ is multiple of $\mathbf{y}$.

11. Suppose $\mathbf{x} \in \mathbb{C}^2$. Give an algorithm for determining a unitary matrix of the form,

$$Q = \begin{bmatrix} c & \bar{s} \\ -s & c \end{bmatrix}$$

$c \in \mathbb{R}$, $c^2 + |s|^2 = 1$ and the second component of $Q^H\mathbf{x}$ is zero.

12. Suppose $\mathbf{x}$ and $\mathbf{y}$ are unit vectors in $\mathbb{R}^n$. Give an algorithm using Givens transformation which computes orthogonal Q such that $Q\mathbf{x} = \mathbf{y}$

13. Let $A \in \mathbb{R}^{m\times n}$, $m > n$. The QR factorization of A is given by $A = QR$, where $A \in \mathbb{R}^{m\times m}$ is an orthonormal matrix and

$$R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$$

where $\tilde{R}$ is an $n \times n$ upper triangular matrix. Let $\mathrm{rank}(A) = p$ and assume that $\tilde{R}$ is arranged so that $|r_{11}| \geq |r_{22}| \geq \ldots \geq |r_{nn}|$. Find an orthonormal basis for $\mathcal{R}(A)$ and $\mathcal{R}(A)_\perp$. Identify the shape of $\tilde{R}$ when $p \leq n$. Using the QR decomposition, find an orthonormal basis for $\mathcal{N}(A)$.

14. Let $\mathbf{x} \in \mathbb{R}^n$ and let H be a Householder matrix such that $H\mathbf{x} = \pm\|\mathbf{x}\|_2\mathbf{e}_1$. Let $G_{1,2}, G_{1,3}, \ldots, G_{n-1,n}$ be Givens rotations, and let $Q = G_{1,2}G_{1,3}\cdots G_{n-1,n}$. Suppose $Q\mathbf{x} = \pm\|\mathbf{x}\|_2\mathbf{e}_1$. Must $Q = H$? (Give a proof or a counterexample).