# E2 212: Homework - 6

## 1 Topics

- Least-squares
  - Using QR: full-rank and rank-deficient cases
  - Using the complete orthogonal decomposition
  - Constrained LS
  - Total LS
  - Iterative LS
- The Jordan canonical form

Note: Most of the problems below are from Golub and Van Loan, Horn and Johnson, or David Lewis' books.

## 2 Problems

1. Show that if
$$A = \begin{bmatrix} R & w \\ 0 & v \end{bmatrix} \text{ and } b = \begin{bmatrix} c \\ d \end{bmatrix},$$
where $R$ is a $k \times k$ block, $w \in \mathbb{R}^k$, $v \in \mathbb{R}^{m-k \times n-k}$, $c \in \mathbb{R}^k$, $d \in \mathbb{R}^{m-k}$ and the zero is a block of appropriate dimensions, and if $A$ has full column rank, then $\min \|Ax - b\|_2^2 = \|d\|_2^2 - (v^T d / \|v\|_2)^2$.

2. Suppose $A \in \mathbb{R}^{n \times n}$ and $D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$. Show how to construct an orthogonal $Q$ such that $Q^T A - DQ^T = R$ is upper triangular.

3. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and that $r = b - Ax$ where $r, b, x \in \mathbb{R}^n$ and $x \neq 0$. Show how to compute a symmetric $E \in \mathbb{R}^{n \times n}$ with minimal Frobenius norm so that $(A + E)x = b$. (Hint: Use the QR factorization of $[x, r]$ and note that $Ex = r \Rightarrow (Q^T E Q)(Q^T x) = Q^T R$.)

4. Show that if
$$A = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix},$$
where $T \in \mathbb{R}^{r \times r}, S \in \mathbb{R}^{r \times n-r}$, $r = \text{rank}(A)$ and $T$ is nonsingular, then
$$X = \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$
satisfies $AXA = A$ and $(AX)^T = AX$. We say that $X$ is a $(1, 3)$ pseudoinverse of $A$. Show that for general $A$, $x_B = Xb$ where $X$ is a $(1, 3)$ pseudoinverse of $A$.

5. Solve
$$\min_{\|\mathbf{x}\|_2 = 1} \left\| \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right\|_2$$

6. Let $Y = [y_1, \ldots, y_k] \in \mathbb{R}^{m\times k}$ be such that

$$Y^T Y = \mathrm{diag}(d_1^2, d_2^2, \ldots, d_k^2), \quad d_1 \geq d_2 \geq \cdots \geq d_k > 0.$$

Show that if $Y = QR$ is the QR factorization of $Y$, then $R$ is diagonal with $|r_{ii}| = d_i$.

7. Prove the total least squares theorem stated in class. That is, let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, with $m \geq n+1$. Let $C = [A \quad \mathbf{b}]$ have the economy-size SVD $C = U\Sigma V^T$, where the diagonal elements of $\Sigma \in \mathbb{R}^{n+1\times n+1}$ are $\sigma_1, \sigma_2, \ldots, \sigma_{n+1}$, $U = [U_1 \quad \mathbf{u}_2]$ where $U_1$ represents the first $n$ columns of $U \in \mathbb{R}^{m\times n+1}$, and

$$V = \begin{bmatrix} V_{11} & \mathbf{v}_{12} \\ \mathbf{v}_{21} & v_{22} \end{bmatrix} \in \mathbb{R}^{n+1\times n+1}$$

where $V_{11}$ is $n\times n$, $v_{22} \in \mathbb{R}$ and $\mathbf{v}_{12}$ and $\mathbf{v}_{21}$ have appropriate dimensions. Also, let $\Sigma_1$ denote the first $n \times n$ block of $\Sigma$.

If $\sigma_n(A) > \sigma_{n+1}(C)$, then the matrix

$$[E_0 \quad \mathbf{r}_0] = -\mathbf{u}_2 \sigma_{n+1}(C)[\mathbf{v}_{12}^T \quad v_{22}]$$

solves

$$\min_{\mathcal{R}(\mathbf{b}+\mathbf{r})\subset\mathcal{R}(A+E)} \|[E \quad \mathbf{r}]\|_2 .$$

Also show that $\mathbf{x}_{TLS} = \mathbf{v}_{12} v_{22}^{-1}$ exists and is the unique solution to $(A+E_0)\mathbf{x} = \mathbf{b} + \mathbf{r}_0$. (Hint: see Golub & Van Loan)

8. Show that the eigenvalues of $M = -L^{-1}U$ (where $L$ and $U$ lower and strictly upper triangular matrices, respectively, as defined as in class for the Gauss-Seidel iterations), are the solutions to the the equation $\det(\lambda L + U) = 0$.

9. Show that if $A \in \mathbb{R}^{n\times n}$ is strictly diagonal dominant, i.e., if $|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|$ for each $i = 1, 2, \ldots, n$, show that the Gauss-Seidel iteration for $A\mathbf{x} = \mathbf{b}$ will always converge.

10. Let $A = \begin{bmatrix} 11 & 1 \\ 10 & 1 \end{bmatrix}$. Using the notation in class, show that $\rho(M) = 0.909$ where $\rho(M)$ is the largest eigenvalue of $M \triangleq -L^{-1}U$, and $A = L + U$. Also show that $\rho(M_\omega) = 0.537$ for $\omega = (11 - \sqrt{11})/5$, and that this is the minimum value of $\rho(M_\omega)$ over all possible $0 \neq \omega \in \mathbb{R}$. (Hint: the characteristic polynomial of $M_\omega$ is quadratic, and the minimum value of $\rho(M_\omega)$ occurs when this quadratic has equal roots.

11. Show that any $n \times n$ matrix $A$ is similar to its transpose.

12. Determine the Jordan canonical form for

$$A = \begin{pmatrix} -2 & -1 & -3 \\ 4 & 3 & 3 \\ -2 & 1 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 6 & -1 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

13. If an $n \times n$ matrix has trace 0 and rank 1, show that it is nilpotent.

14. Let $A \in \mathbb{R}^{n\times n}$ have Jordan canonical form $J$. Let $\lambda$ be an eigenvalue of $A$, $a_\lambda$ be the algebraic multiplicity of $\lambda$, $k$ be the size of the largest Jordan block corresponding to $\lambda$, $N_i$ be the number of Jordan blocks of size $i$ corresponding to $\lambda$, $i = 1, 2, \ldots, k$, and $r_j$ be the rank$(A - \lambda I)^j$, $j = 1, 2, \ldots$. Show that

    (a) $a_\lambda = N_1 + 2N_2 + \ldots + kN_k$.

    (b) $r_j = n - a_\lambda, j \geq k$.

    (c) $r_j = n - a_\lambda + \sum_{i=m+1}^{n}(i - m)N_i$, $j = 1, 2, \ldots, k - 1$.