E2 212: Homework - 1

1 Topics

- Basic definitions
- Fundamental subspaces of a matrix, rank

2 Problems

1. Suppose a subspace V is known to have dimension k. Prove that:
   (a) any k independent vectors in V form a basis;
   (b) any k vectors that span V form a basis.

2. Prove that if V and W are three-dimensional subspaces of $\mathbb{R}^5$, then V and W must have a non-zero vector in common.

3. Suppose V is a vector space of dimension 7 and W is a subspace of dimension 4. True or False:
   (a) Every basis for W can be extended to a basis for V by adding three more vectors;
   (b) Every basis for V can be reduced to a basis for W by removing three vectors.

4. Suppose $\mathbf{A}$ is a $m \times n$ matrix. Give the conditions on the rank of $\mathbf{A}$ such that left and right inverses exist, respectively. Can the left and right inverses of a matrix exist simultaneously ?

5. If the product of two matrices is the zero matrix, $\mathbf{AB} = 0$, show that the column space of $\mathbf{B}$ is contained in the null space of $\mathbf{A}$.

6. If $\mathbf{Ax} = \mathbf{b}$ has at least one solution, show that the only solution to $\mathbf{A}^T\mathbf{y} = 0$ is $\mathbf{y} = 0$.

7. Find a basis for each of the four fundamental subspaces of:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Prove that if S is a subspace then $S = (S^{\perp})^{\perp}$. Also, prove that if S is a finite set, $S^{\perp}$ is a subspace. What does $(S^{\perp})^{\perp}$ equal, if S is a finite set ?

9. If V and W are orthogonal subspaces, show that the only vector they have in common is the zero vector.

10. (Fredlholm's alternative) For any $\mathbf{A}$ and $\mathbf{b}$, one and only one of the following systems has a solution: (1) $\mathbf{Ax} = \mathbf{b}$ (2) $\{\mathbf{A}^T\mathbf{y} = 0,\ \mathbf{y}^T\mathbf{b} \neq 0\}$. Show that it is contradictory for (1) and (2) both to have solutions.

11. Show that an orthogonal matrix (square matrix with orthonormal columns) which is also upper triangular must be diagonal.

12. (Rank Inequalities)

    (a) (Sylvester's rank inequality): If $\mathbf{A}$ is $m \times k$ and $\mathbf{B}$ is $k \times n$, then prove that: $\mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{B}) - k \leq \mathrm{rank}(\mathbf{AB}) \leq \min\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B})\}$

    (b) Prove that $\mathrm{rank}(\mathbf{A} + \mathbf{B}) \leq \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{B})$.

    (c) Suppose $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times m$, with $n < m$. Prove that their product $\mathbf{AB}$ is singular.

    (d) Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mathbf{B} \in \mathbb{C}^{n \times n}$. Prove that if $\mathbf{AB}$ is invertible, then both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

    (e) Let $\mathbf{A}$ is $m \times m$ non-singular matrix and let $\mathbf{B}$ be a $m \times n$ matrix. Prove that $\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{B})$.

13. Prove that any set of nonzero mutually orthogonal vectors are linearly independent.

14. Prove that row rank of a given matrix equals its column rank.