E2 212: Homework - 2

1 Topics

- **Determinants**
- **Inner products**
- **Norms**
- **Gram-Schmidt**

2 Problems

1. Show that $\det(AB) = \det(A)\det(B)$.
2. Prove that $\det(A) = \det(A^T)$.
3. Prove that the matrix $A \in \mathbb{R}^{n\times n}$ is singular if and only if $\det(A) = 0$.
4. Let $A \in \mathbb{R}^{n\times n}$. Prove that the determinant of B obtained by
   (a) interchanging rows i and j is $-\det(A)$.
   (b) multiplying row i by $\alpha \neq 0$ is $\alpha \det(A)$.
5. For any real $n \times n$ matrix $A = [a_{ij}]$, $i, j = 1, \ldots, n$ and $a_{ij} \in \mathbb{R}$, prove the following inequality:
$$(\det(A))^2 \leq \prod_{i=1}^{n} b_{ii}.$$
   In the above, b_{ii} is the i-th diagonal entry in $A^T A$.
6. Show that $\det(I + AA^T) = \det(I + A^T A)$.
7. Let V be an inner product space. Prove that for all $\mathbf{x}, \mathbf{y} \in V$,
$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\| \|\mathbf{y}\|,$$
   where the norm is defined as $\| * \| \triangleq \sqrt{\langle *, * \rangle}$. Equality holds if and only if $\mathbf{y} = \alpha\mathbf{x}$ for $\alpha = \langle \mathbf{x}, \mathbf{y} \rangle / \|\mathbf{x}\|$.
8. Prove the following:
   (a) (Minkowski inequality): For every $p > 0$
$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

(b) (Holder's inequality) If $p > 0$ and $q > 0$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

9. Let V be a finite dimensional vector space. The norm $\| * \|_A$ is said to be equivalent to the norm $\| * \|_B$ denoted $\| * \|_A \sim \| * \|_B$ if for all $\mathbf{x} \in V$, there exists $0 < K_1 \le K_2$ such that

$$K_1 \|\mathbf{x}\|_A \le \|\mathbf{x}\|_B \le K_2 \|\mathbf{x}\|_B.$$

Show that the relation $\sim$ is an equivalence relation, i.e., prove the following:

(a) $\| * \|_A \sim \| * \|_A$.

(b) $\| * \|_A \sim \| * \|_B$, then $\| * \|_B \sim \| * \|_A$.

(c) If $\| * \|_A \sim \| * \|_B$ and $\| * \|_B \sim \| * \|_C$, then $\| * \|_A \sim \| * \|_C$.

10. Let $(V, \langle *, * \rangle)$ be a linear inner product space of dimension n. For any fixed $\mathbf{x} \in V$, find the dimension of the subspace $W \triangleq \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle = 0\}$.

11. Show that, if $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} \|\mathbf{x}\|_2 &\le \|\mathbf{x}\|_1 \le \sqrt{n} \|\mathbf{x}\|_2 \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_2 \le \sqrt{n} \|\mathbf{x}\|_\infty \\ \|\mathbf{x}\|_\infty &\le \|\mathbf{x}\|_1 \le n \|\mathbf{x}\|_\infty \end{aligned}$$

When is the equality attained?

12. Let $\| \cdot \|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m \times n}$. Show that if $\text{rank}(A) = n$, then $\|\mathbf{x}\|_A \triangleq \|A\mathbf{x}\|$ is a vector norm on $\mathbb{R}^n$.

13. Show that the Frobenius norm, defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2},$$

and the p-norm, defined by

$$\|A\|_p = \max_{\mathbf{x} \ne \mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \quad p \ge 1$$

are matrix norms.

14. For a real inner product space $(V, \langle ., . \rangle)$ with the norm induced by the inner product $(\| * \|^2 = \langle *, * \rangle)$, prove that

$$\langle \mathbf{x}, \mathbf{y} \rangle \le \frac{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}{2}.$$

15. A matrix $A \in \mathbb{C}^{n \times n}$ is said to be normal if $AA^H = A^H A$. Prove that if A is normal, then $\text{Range}(A) \perp \text{Null}(A)$.

16. Apply Gram-Schmidt procedure to obtain an orthonormal set for the following set of vectors:

(a) $\{(-1, 0, 1), (-1, -1, 0), (0, 0, 1)\} \subseteq \mathbb{R}^3$.

(b) $\{(1, -1, 1, -1), (5, 1, 1, 1), (2, 3, 1, -1)\} \subseteq \mathbb{R}^4$.