E2 212: Homework - 3

1 Topics

- Matrix Norms
- Errors in solutions of linear equations and condition numbers
- Eigenvalues and Eigenvectors

2 Problems

Notation: M_n denotes an $n \times n$ matrix over a field of complex (or real) numbers, i.e. $\mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$). Note that "triple-bar" norms $|||\cdot|||$ denote vector induced matrix norms while "double-bar" norms $\|\cdot\|$ denote vector norms (possibly, on matrices).

1. Show that, for any $\mathbf{A} \in M_n$, the series $\sum_{k=0}^{\infty} a_k \mathbf{A}^k$ converges if there is a matrix norm $|||\ .\ |||$ on M_n such that the numerical series $\sum_{k=0}^{\infty} |a_k|\ |||A|||^k$ converges. (Hint: What does convergence of a series mean?)

2. If $\mathbf{A}, \mathbf{B} \in M_n$, if $\mathbf{A}$ is invertible, and if $\mathbf{A}+\mathbf{B}$ is singular, show that $|||\mathbf{B}||| \geqslant 1/|||\mathbf{A}^{-1}|||$ for any matrix norm $|||\ .\ |||$. Thus there is an intrinsic limit to how well a non-singular matrix can be approximated by a singular one. (Hint: $\mathbf{A}+\mathbf{B} = \mathbf{A}(\mathbf{I}+\mathbf{A}^{-1}\mathbf{B})$.)

3. Show that if $\mathbf{B}$ is an idempotent matrix then $|||\mathbf{B}||| \geqslant 1$ for any matrix norm $|||\ .\ |||$.

4. Give an example of a vector norm on matrices for which $\|\mathbf{I}\| < 1$.

5. Show that:
$$|||\mathbf{A}|||_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \max_{\|\mathbf{x}\|_2=\|\mathbf{y}\|_2=1} |\mathbf{y}^H \mathbf{A}\mathbf{x}|$$

6. Prove the following:
   (a) $|||\mathbf{A}|||_1 \leqslant \|\mathbf{A}\|_1 \leqslant n|||\mathbf{A}|||_\infty$. Here $|||\ .\ |||_1$ denotes the operator (matrix) norm induced by l_1-vector norm and $\|\ .\ \|_1$ is the l_1 vector norm for matrices (sum of absolute values of entries of matrix).
   (b) $|||\mathbf{A}|||_1 \leqslant \sqrt{n}\ \|\mathbf{A}\|_2$

7. Show that $\kappa(\mathbf{AB}) \leqslant \kappa(\mathbf{A})\kappa(\mathbf{B})$ always, where $\kappa(\ .\ )$ is the condition number for a given matrix. Is $\kappa(\ .\ )$ a matrix or a vector norm?

8. Let $\mathbf{A}$ be a unitary matrix. Prove that $\kappa(\mathbf{A}) = 1$ with respect to the spectral norm.

   Hint: You do not need eigenvalues for this problem! Use the following properties of unitary matrices. If $\mathbf{A}$ is unitary matrix, then:
   (a) $\mathbf{A}\mathbf{A}^H = \mathbf{I}$, i.e., it's hermitian is it's inverse.
   (b) $\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}^H\mathbf{x}\|_2 = \|\mathbf{x}\|_2$, i.e., it preserves the Euclidean norm.

9. Show that for any square matrix $\mathbf{A}$, the set $E_\lambda \triangleq \{\mathbf{v} \in \mathbb{R}^n : \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}$ for any $\lambda \in \mathbb{R}$, is a subspace.

10. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Show that $\mathbf{AB}$ and $\mathbf{BA}$ have exactly the same eigenvalues.

11. Give a closed-form solution for $\mathbf{x}$ in the system of equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is full rank, square and symmetric, in terms of the eigendecomposition of $\mathbf{A}$.

12. The so-called power method for calculating the largest eigenvalue/vector pair of a square symmetric matrix $\mathbf{A}$ is described as follows:
Initialize: set $\mathbf{x}_0$ to an arbitrary value. For $i = 1, 2, 3, \ldots$

$$\mathbf{z} = \mathbf{A}\mathbf{x}_{i-1}$$

$$\mathbf{x}_i = \frac{\mathbf{z}}{\|\mathbf{z}\|_2}$$

As $i \to \infty$, $\mathbf{x}_i$ converges to the maximum eigenvector.

    (a) Prove that the above algorithm converges. *Hint:* express $\mathbf{x}_0$ using the eigenvectors of $\mathbf{A}$ as a basis.

    (b) Modify the algorithm to find the smallest eigenvalue/vector.

    (c) Explain what happens when the largest eigenvalue is not distinct.

13. Derive an analytical expression for the eigendecomposition of a square symmetric rank-one matrix.