# E2 212: Homework - 4

## 1 Topics

- Eigenvalues and Eigenvectors
- Schur's theorem
- Unitary equivalence
- Normal matrices
- $QR$ decomposition and Jordan form

## 2 Problems

1. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n \times n}$. Then, prove the following:
   (a) $\text{tr } \mathbf{A} = \sum_{i=1}^{n} \lambda_i$.
   (b) $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

2. Let the eigenvectors and eigenvalues of the matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ be $\mathbf{v}_i$ and $\lambda_i$ such that $\lambda_i \neq \lambda_j$ for all $j \neq i$, $i = 1, 2, \ldots, r$. Then, prove that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a linearly independent set.

3. Let the subspace $U \subseteq \mathbb{C}^{n \times n}$ be an invariant subspace under the matrix transformation $\mathbf{A} \in \mathbb{C}^{n \times n}$, i.e., $\mathbf{Au} \subseteq U$ for all $\mathbf{u} \in U$. Then, prove the following
   (a) There exists a vector $\mathbf{u} \in U$ and $\lambda \in \mathbb{C}$ such that $\mathbf{Au} = \lambda \mathbf{u}$.
   (b) If $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$ are eigenvectors of $A$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\dim(U) \geq k$.
   (c) The subspace $U^{\perp} \triangleq \{\mathbf{v} \in V : \langle \mathbf{u}, \mathbf{v} \rangle = 0, \text{for all } \mathbf{u} \in U\}$ is also an invariant subspace with respect $\mathbf{A}$.

4. Let $\sigma(\mathbf{A})$ denote the spectrum of the matrix $\mathbf{A}$. Show that for a triangular matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ with entries $a_{ij}$, $\sigma(\mathbf{A}) = \bigcup_{i=1}^{n} \{a_{ii}\}$.

5. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ has $n$ distinct eigenvalues in $\mathbb{C}$. Then, prove that there exists an invertible matrix $U \in \mathbb{C}^{n \times n}$ and a diagonal matrix $D$ such that

$$\exp\{\mathbf{A}\} = UDU^{-1}.$$

6. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$. Prove that if $(\mathbf{A} - \lambda I_n)^j \mathbf{u} = 0$ for some $j \geq 1$, and $\mathbf{u} \in \mathbb{C}^n$, then $(\mathbf{A} - \lambda I_n)^n \mathbf{u} = 0$.

7. Let $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n \times n}$. Then,

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2 .$$

8. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$, and $\beta \triangleq \max\{|a_{ij}| : i,j = 1,2,\ldots,n\}$. Then,

$$|\det \mathbf{A}| \leq \beta^n n^{n/2}.$$

9. Prove that for every matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, and for every $\epsilon > 0$, there exists a matrix $\mathbf{A}_\epsilon$ such that $\mathbf{A}_\epsilon$ is equivalent to a diagonal matrix, and $\|\mathbf{A} - \mathbf{A}_\epsilon\|_F < \epsilon$. *Hint:* Use Schur's theorem.

10. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be given, and suppose $\mathbf{A}$ and $\mathbf{B}$ are simultaneously similar to upper triangular matrices: that is, $\mathbf{S}^{-1}\mathbf{AS}$ and $\mathbf{S}^{-1}\mathbf{BS}$ are both upper triangular for some nonsingular $\mathbf{S}$. Show that every eigenvalue of $\mathbf{AB} - \mathbf{BA}$ must be zero.

11. If $\mathbf{A} \in \mathbb{C}^{n\times n}$, show that rank of $\mathbf{A}$ is not less than the number of nonzero eigenvalues of $\mathbf{A}$.

12. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. Show that any matrix that commutes with $A$ also commutes with $\mathbf{A}^{-1}$.

13. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is normal, and if $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to distinct eigenvalues, show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

14. Show that a normal matrix is unitary if and only if all its eigenvalues have absolute value 1.

15. Show that a given matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is normal if and only if

$$(\mathbf{Ax})^H(\mathbf{Ay}) = (\mathbf{A}^H\mathbf{x})^H(\mathbf{A}^H\mathbf{y}).$$

16. Let $n_1, n_2, \ldots, n_k$ be given positive integers and let $A_j \in M_{n_j}$, $j = 1,2,\ldots,k$. Show that the direct sum $\mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_k$ is normal if and only if each $\mathbf{A}_j$ is normal.

17. Show that each $\mathbf{A}_k$ produced by the $QR$ algorithm is unitarily equivalent to $\mathbf{A}_0$, $k = 1,2,\ldots$.

18. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be given. Define

$$K_\lambda \triangleq \{\mathbf{x} \in \mathbb{C}^n : (\mathbf{A} - \lambda I)^p\mathbf{x} = 0 \text{ for some integer } p > 0\}.$$

Prove that

(a) If $\lambda$ is an eigenvalue of $\mathbf{A}$, then $K_\lambda$ is a $\mathbf{A}-$invariant subspace of $\mathbb{C}^n$, i.e., $\mathbf{Ay} \subseteq K_\lambda$ for all $\mathbf{y} \in K_\lambda$.

(b) If $\lambda$ is an eigenvalue of $\mathbf{A}$ with multiplicity $m$, then $\dim(K_\lambda) \leq m$.

19. Find an invertible matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{AU}$ is in Jordan form when

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix},$$

where $i = \sqrt{-1}$.

20. Find the Jordan form of the following matrices

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 2 \\ -i & 0 & 1 \end{bmatrix}.$$