E2 212: Homework - 7

1 Topics

- **Congruent matrices**
- **Geršgorin disc theorem**
- **Perturbation of eigenvalues**

2 Problems

1. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ are Hermitian matrices with all the eigenvalues of $\mathbf{A}-\mathbf{B}$ being nonnegative, then prove that $\lambda_i(\mathbf{A}) \geq \lambda_i(\mathbf{B})$ for all $i = 1, 2, \ldots, n$, where the eigenvalues of both matrices are arranged in increasing order, as usual.

2. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be Hermitian. Show that $\mathbf{A}$ is congruent to the identity matrix if and only if all the eigenvalues of $\mathbf{A}$ are positive.

3. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ be Hermitian matrices and assume that $\mathbf{A}$ is positive definite. Then prove that

   (a) There exists a non singular matrix $\mathbf{S} \in \mathbb{C}^{n\times n}$ and a real diagonal $\mathbf{D}$ such that $\mathbf{S}^H\mathbf{A}\mathbf{S} = \mathbf{I}$, and $\mathbf{S}^H\mathbf{B}\mathbf{S} = \mathbf{D}$.

   (b) The diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$, and the columns of $\mathbf{S}$ are the eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$ corresponding to these eigenvalues.

4. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ have entries $a_{ij} = 1$ for all $i \neq j$, and $a_{ii} = n$ for each i. Show that $\mathbf{A}$ is invertible.

5. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ and the Geršgorin discs for the rows of $\mathbf{A}$ are mutually disjoint, then, show that all of the eigenvalues of $\mathbf{A}$ must be real.

6. Consider the following iterative algorithm for solving the n-by-n system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{y}$, where $\mathbf{A}$ and $\mathbf{y}$ are given:

   - Define $\mathbf{B} = I - \mathbf{A}$ and rewrite the system as $\mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{y}$.
   - Choose an initial approximation $\mathbf{x}^{(0)}$ to the solution in any way you wish.
   - For $m = 0, 1, 2, \ldots$ calculate $\mathbf{x}^{(m+1)} = \mathbf{B}\mathbf{x}^{(m)} + y$.

   (a) Denote by $\epsilon^{(m)} = \mathbf{x}^{(m)} - \mathbf{x}$ the error in the mth approximation to the solution, and show that

   $$\epsilon^{(m)} = \mathbf{B}^m(\mathbf{x}^{(0)} - \mathbf{x}).$$

   (b) Conclude that if $\rho(I - \mathbf{A}) < 1$, then this algorithm works in the sense that $\mathbf{x}^{(m)} \to \mathbf{x}$ as $m \to \infty$ regardless of the choice of the initial approximation $\mathbf{x}^{(0)}$.

   (c) Use Geršgorin theorem to give a simple explicit condition on $\mathbf{A}$ that is sufficient for this algorithm to work.

7. Without computing the eigenvalues, decide how many are positive, negative and zero for the following matrix ? Why ?

$$\mathbf{A} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & n-1 \\ 1 & \dots & n-1 & n \end{pmatrix}.$$

8. Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a real matrix whose n Geršgorin discs are all mutually disjoint. Show that all the eigenvalues of $\mathbf{A}$ are real.

9. Suppose that $\mathbf{A}$ is idempotent ($\mathbf{A}^2 = \mathbf{A}$), but $\mathbf{A} \neq I$. Show that $\mathbf{A}$ cannot be strictly diagonally dominant.

10. Let $\mathbf{A} = [a_{ij}]$ be a $n \times n$ matrix. Show that

$$\rho(\mathbf{A}) \leq \min_{p_1,p_2,\dots,p_n>0} \max_{1\leq i\leq n} \frac{1}{p_i} \sum_{j=1}^{n} p_j \left|a_{ij}\right|.$$

Hint: Since the eigenvalues do not change under similarity transformations, we can tighten the Geršgorin regions by considering similarity transformations.

11. If $\mathbf{A} \in \mathbb{C}^{n\times n}$, and $\rho(\mathbf{A})$ represents the spectral radius of $\mathbf{A}$. Using Geršgorin theorem, prove that

    (a) $\rho(\mathbf{A}) \leq \max_{1<i<n} \sum_{j=1}^{n} |a_{ij}|$

    (b) $\rho(\mathbf{A}) \leq \max_{1<j<n} \sum_{i=1}^{n} |a_{ij}|$

12. Let λ be a simple eigenvalue (eigenvalue with algebraic multiplicity 1) of an $n \times n$ matrix $\mathbf{A}$. Let $s(\lambda) = \left|\mathbf{y}^H\mathbf{x}\right|$ where $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}^H\mathbf{A} = \lambda\mathbf{y}^H$ and $\mathbf{x}$ and $\mathbf{y}$ have unit Euclidean norm. Prove the following:

    (a) $S(\lambda) \leq 1$.

    (b) $S(\lambda) \neq 0$.