# E2 212: Homework - 8

## 1 Topics

- **SVD**
- **Least squares**

Note: Most of the problems below are from Golub and Van Loan, Horn and Johnson, or David Lewis' books.

## 2 Problems

1. Show that $\sigma_1, \ldots, \sigma_r$ are the non-zero singular values of the matrix $\mathbf{A}$ iff $\{\sigma_1, \ldots, \sigma_r, -\sigma_1, \ldots, -\sigma_r\}$ are the non-zero eigenvalues of the matrix

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{pmatrix}.$$

2. Let $\mathbf{A}$ be an $m \times n$ real matrix with the "economy" singular value decomposition $\mathbf{A} = \mathbf{U\Sigma V}^T$. Show that the unitary matrix $\mathbf{B} = \mathbf{UV}^T$ is the closest unitary matrix to $\mathbf{A}$, i.e., for any unitary matrix $\mathbf{P}$, $\|\mathbf{B} - \mathbf{A}\| \leq \|\mathbf{P} - \mathbf{A}\|$.

3. Let $\mathbf{A}$ be an $m \times n$ matrix with rank $r$ and whose non-zero singular values are $\sigma_1, \sigma_2, \ldots, \sigma_r$. Show that the Euclidean (Frobenius) norm of $A$ is given by

$$\|\mathbf{A}\| = \sqrt{\sum_{i=1}^{r} \sigma_i^2}.$$

4. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, and $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\|\mathbf{Ax} - \mathbf{b}\|^2 = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j - b_i \right)^2.$$

   Show that the equations $\partial f / \partial \mathbf{x}_i = 0$, $i = 1, \ldots, n$, where $f(\mathbf{x}) \triangleq \|\mathbf{Ax} - \mathbf{b}\|^2$, are equivalent to the normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

5. Let $\hat{\mathbf{w}}$ be the solution to

$$\min_{\mathbf{w}} \|\mathbf{y} - H\mathbf{w}\|^2, \tag{1}$$

   and let $\hat{\mathbf{y}} \triangleq H\hat{\mathbf{w}}$. Show that

   (a) The fundamental orthogonality principle holds, i.e., $\hat{\mathbf{w}}$ is a solution if, and only if, the residual $\tilde{\mathbf{y}} \triangleq \mathbf{y} - H\hat{\mathbf{w}}$ is orthogonal to $\mathcal{R}(H)$.

(b) The norms of $\{\mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}\}$ satisfy the relation $\|\mathbf{y}\|^2 = \|\hat{\mathbf{y}}\|^2 + \|\tilde{\mathbf{y}}\|^2$.

6. Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{w} \\ 0 & \mathbf{v} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

where $\mathbf{R}$ is a $k \times k$ block, $\mathbf{c}, \mathbf{w} \in \mathbb{R}^k$, $\mathbf{v}, \mathbf{d} \in \mathbb{R}^{m-k}$, and the zero is a block of appropriate dimension. If $\mathbf{A} \in \mathbb{R}^{m \times (k+1)}$ has full column rank, then show that

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \|\mathbf{d}\|_2^2 - (\mathbf{v}^T\mathbf{d}/\|\mathbf{v}\|_2)^2.$$

7. Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a set of vectors $\mathbf{x}_i \in \mathbb{R}^m$, $\mathbf{y}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, k$,

   (a) Find a set of coefficients $a_i$'s such that $\|\mathbf{A} - \sum_{i=1}^{k} a_i \mathbf{x}_i \mathbf{y}_i^T\|_F^2$ is minimized.

   (b) What are the set of $\mathbf{x}_i, \mathbf{y}_i$ that minimize the minimum in (a) for $k < \min(m, n)$?

8. Given the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{A}$ being a tall matrix, what is $\mathbf{P}$ such that $[\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{P}\mathbf{b}\|]$ is minimized?