E2 212: Homework - 9

1 Topics

- **Nonnegative Matrices**

Note: The problems below are from Horn and Johnson.

2 Problems

1. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has a spectral radius 1, but that $\mathbf{A}^m$ is unbounded as $m \to \infty$.

2. Consider the matrix
$$\mathbf{A}_\epsilon = \begin{bmatrix} \frac{1}{1+\epsilon} & \frac{1}{1+\epsilon} \\ \frac{\epsilon^2}{1+\epsilon} & \frac{1}{1+\epsilon} \end{bmatrix}, \quad \epsilon > 0.$$
   (a) Show that $\lambda_2 = 1$ is a simple eigenvalue of $\mathbf{A}_\epsilon$, that $\rho(\mathbf{A}_\epsilon) = \lambda_2 = 1$, and $|\lambda_1| < 1$.

   (b) Show that
$$\mathbf{x} = \frac{1}{1+\epsilon}\begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \text{ and } \mathbf{y} = \frac{1+\epsilon}{2\epsilon}\begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$$
   are eigenvectors of $\mathbf{A}_\epsilon$ and $\mathbf{A}_\epsilon^T$, respectively, corresponding to the eigenvalue $\lambda = 1$.

   (c) Calculate $\mathbf{A}_\epsilon^m$ explicitly, $m = 1, 2, \ldots$.

   (d) Show that
$$\lim_{m\to\infty} \mathbf{A}_\epsilon^m = \frac{1}{2}\begin{bmatrix} 1 & \epsilon^{-1} \\ \epsilon & 1 \end{bmatrix}.$$
   (e) Calculate $\mathbf{x}\mathbf{y}^T$ and comment.

   (f) What happens if $\epsilon \to 0$? Hint: Set $\mathbf{B}_\epsilon = (1+\epsilon)\mathbf{A}_\epsilon$ and then diagonalize $\mathbf{B}$.

3. Given an example of a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A} \geq 0$, $\mathbf{A}$ not positive, and $\mathbf{A}^2 > 0$. Show that $\rho(A) > 0$ for all such matrices.

4. If $0 \leq \mathbf{A} \leq \mathbf{B} \in \mathbb{C}^{n\times n}$, show that $\rho(A) \leq \rho(B)$. Also show that $\rho(\mathbf{A}) \geq \max_{i=1,\ldots,n} a_{ii}$.

5. If $\mathbf{A} \geq 0$ has a positive eigenvector, show that $\mathbf{A}$ is similar to a non-negative matrix whose row sums are constant. What is this constant?

6. If $\mathbf{A} > 0$, and if there is some $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x} \geq 0, \mathbf{x} \neq 0$, and $\mathbf{A}x = \lambda\mathbf{x}$, show that $\mathbf{x}$ is a multiple of the Perron vector of $\mathbf{A}$ and that $\lambda = \rho(\mathbf{A})$.

7. If $\mathbf{A} > 0$, if $\mathbf{x}$ is the Perron vector of $\mathbf{A}$, and if $\mathbf{z}$ is the Perron vector of $\mathbf{A}^T$, show that $\mathbf{x}^T\mathbf{z} > 0$.

8. In the general intercity migration problem discussed in class, when the number of cities n is > 2, if all $a_{ij} > 0$, what is the asymptotic behavior of the population as the number of days m goes to ∞? Justify your answer.

9. Let $0 \leq \mathbf{A} \in \mathbb{C}^{n \times n}$, $0 \leq \mathbf{x} \in \mathbb{C}^n$, and $\mathbf{x} \neq 0$. If $\mathbf{Ax} \geq \alpha \mathbf{x}$ for some $\alpha \in \mathbb{R}$, then show that $\rho(\mathbf{A}) \geq \alpha$.

10. Let $\mathbf{A} \geq 0$. Then show that the following statements are equivalent:

    (a) $\mathbf{A}$ is irreducible

    (b) $(\mathbf{I} + \mathbf{A})^{n-1} > 0$

    (c) $\mathbf{A}^T$ is irreducible.

11. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ (including multiplicities). Then show that $\lambda_1 + 1, \ldots, \lambda_n + 1$ are the eigenvalues of $\mathbf{I} + \mathbf{A}$ and $\rho(\mathbf{I} + \mathbf{A}) \leq 1 + \rho(\mathbf{A})$. Also, show that if $\mathbf{A} \geq 0$, then $\rho(\mathbf{I} + \mathbf{A}) = 1 + \rho(\mathbf{A})$. Finally, explain why the following argument is incorrect: if λ is an eigenvalue of $\mathbf{A}$, then there is some vector $\mathbf{x} \neq 0$ such that $\mathbf{Ax} = \lambda \mathbf{x}$. But then $(\mathbf{A} + \mathbf{I})\mathbf{x} = (\lambda + 1)\mathbf{x}$, so $\lambda + 1$ is an eigenvalue of $\mathbf{A} + \mathbf{I}$.

12. Let $n > 1$ be a prime number. Show that if $\mathbf{A} \in \mathbb{C}^{n \times n}$ is nonnegative, irreducible, and nonsingular, either $\rho(\mathbf{A})$ is the only eigenvalue of $\mathbf{A}$ of maximum modulus or all the eigenvalues of $\mathbf{A}$ have maximum modulus.

13. Show that the sets of stochastic and doubly stochastic matrices in $\mathbb{C}^{n \times n}$ are compact convex sets.

14. Show that any 2×2 doubly stochastic matrix is symmetric with equal diagonal entries.

15. If a doubly stochastic matrix $\mathbf{A}$ is reducible, show that $\mathbf{A}$ is actually permutation-similar to a matrix of the form $\begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix}$. What can be said about $\mathbf{A}_1$ and $\mathbf{A}_2$.