E2 212: Matrix Theory Fall 2019 – Test 1 Solutions

Questions and Answers

1. **(Linear independence)**

   (a) For what ξ are the vectors $\begin{pmatrix} 1+\xi \\ 1-\xi \end{pmatrix}$ and $\begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}$ on $\mathbb{C}^2$ linearly dependent? (2 points)

   Solution: Let $\mathbf{x}_1 = \begin{pmatrix} 1+\xi \\ 1-\xi \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1-\xi \\ 1+\xi \end{pmatrix}$. Then, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent if and only if the only solution to $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = \mathbf{0}$ is $a_1 = a_2 = 0$. Now, $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = \mathbf{0}$ implies

   $$\begin{aligned} a_1 + a_2 + \xi(a_1 - a_2) &= 0 \\ a_1 + a_2 + \xi(a_2 - a_1) &= 0 \end{aligned}$$

   Subtracting the two equations, we get

   $$2\xi(a_2 - a_1) = 0$$

   and thus either $\xi = 0$ or $a_1 = a_2$. But substituting the latter into the first equation above we get $2a_1 = 0$, which yields $a_1 = a_2 = 0$, i.e., $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. On the other hand, if $\xi = 0$, we get $a_1 + a_2 = 0$ which has nonzero solutions. Thus, the only way $\mathbf{x}_1$ and $\mathbf{x}_2$ can be linearly dependent is if $\xi = 0$.

   (b) Show that $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are linearly independent if and only if $\mathbf{x}+\mathbf{y}$, $\mathbf{y}+\mathbf{z}$ and $\mathbf{z}+\mathbf{x}$ linearly independent. (4 points)

   Solution: Suppose $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are linearly independent. Then the only solution to $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = 0$ is $\alpha = \beta = \gamma = 0$.

   Now consider $a(\mathbf{x}+\mathbf{y}) + b(\mathbf{y}+\mathbf{z}) + c(\mathbf{z}+\mathbf{x}) = 0$. This can be written as $(a + c)\mathbf{x} + (a+b)\mathbf{y} + (b+c)\mathbf{z} = 0$, which implies $a + b = 0$, $a + c = 0$ and $b + c = 0$.

   Subtracting the first two equations and adding it to the third, we get $2b = 0$ or $b = 0$. Substituting this, we get $a = c = 0$.

   Thus $a = b = c = 0$ and hence $\mathbf{x}+\mathbf{y}$, $\mathbf{y}+\mathbf{z}$, and $\mathbf{z}+\mathbf{x}$ are linearly independent.

Contrariwise, if $\mathbf{x}+\mathbf{y}$, $\mathbf{y}+\mathbf{z}$, and $\mathbf{z}+\mathbf{x}$ are linearly independent, then the only solution to $a(\mathbf{x}+\mathbf{y})+b(\mathbf{y}+\mathbf{z})+c(\mathbf{z}+\mathbf{x})=0$ is $a=b=c=0$.

Now consider $2\alpha\mathbf{x}+2\beta\mathbf{y}+2\gamma\mathbf{z}=0$. This can be written as $(\alpha+\beta-\gamma)(\mathbf{x}+\mathbf{y})+(\beta+\gamma-\alpha)(\mathbf{y}+\mathbf{z})+(\gamma+\alpha-\beta)(\mathbf{z}+\mathbf{x})=0$. This implies $\alpha+\beta-\gamma=0$, $\beta+\gamma-\alpha=0$, and $\gamma+\alpha-\beta=0$. Adding the first two equations yields $\beta=0$. Adding the second and third equations yields $\gamma=0$, and adding the first and third equations yields $\alpha=0$. Thus, the only solution is $\alpha=\beta=\gamma=0$, i.e., $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent.

2. **(Rank)**

(a) Let $\mathbf{A}$, $\mathbf{B}\in\mathbb{R}^{m\times n}$. Then, prove that (6 points)

$$|\text{rank}(\mathbf{A})-\text{rank}(\mathbf{B})|\leqslant\text{rank}(\mathbf{A}-\mathbf{B})$$

Solution:
$|\text{rank}(\mathbf{A})-\text{rank}(\mathbf{B})|\leqslant\text{rank}(\mathbf{A}-\mathbf{B})$ is equivalent to

$$-\text{rank}(\mathbf{A}-\mathbf{B})\leqslant\text{rank}(\mathbf{A})-\text{rank}(\mathbf{B})\leqslant\text{rank}(\mathbf{A}-\mathbf{B})$$

Consider $\mathbf{A}=\mathbf{A}+\mathbf{B}-\mathbf{B}$. Since $\text{rank}(\mathbf{A}+\mathbf{B})\leqslant\text{rank}(\mathbf{A})+\text{rank}(\mathbf{B})$,

$$\text{rank}(\mathbf{A})=\text{rank}(\mathbf{A}+\mathbf{B}-\mathbf{B})\leqslant\text{rank}(\mathbf{A}-\mathbf{B})+\text{rank}(\mathbf{B})$$

Similarly $\mathbf{B}=\mathbf{B}+\mathbf{A}-\mathbf{A}$.

$$\text{rank}(\mathbf{B})\leqslant\text{rank}(\mathbf{B}-\mathbf{A})+\text{rank}(\mathbf{A})$$

But $\text{rank}(\mathbf{B}-\mathbf{A})=\text{rank}(\mathbf{A}-\mathbf{B})$, the inequality follows.

(b) If $\mathbf{A},\mathbf{B}\in\mathbb{R}^{n\times n}$ and $\mathbf{AB}=0$, show that (6 points)

$$\text{rank}(\mathbf{A})+\text{rank}(\mathbf{B})\leqslant n$$

Solution: Since $\mathbf{AB}=0$, column space of $\mathbf{B}$ ($\mathcal{R}(\mathbf{B})$) is contained in null space of $\mathbf{A}$ ($\mathcal{N}(\mathbf{A})$). [Consider any $\mathbf{x}\in\mathcal{R}(\mathbf{B})$, it can be shown that $\mathbf{x}\in\mathcal{N}(\mathbf{A})$]. Thus:

$$\begin{aligned}\mathcal{R}(\mathbf{B})&\subseteq\mathcal{N}(\mathbf{A})\\ \implies\dim(\mathcal{R}(\mathbf{B}))&\leqslant\dim(\mathcal{N}(\mathbf{A}))\\ \implies\text{rank}(\mathbf{B})&\leqslant n-\text{rank}(\mathbf{A})\\ \implies\text{rank}(\mathbf{B})+\text{rank}(\mathbf{A})&\leqslant n.\end{aligned}$$

3. **(Skew Symmetric Matrices)** Let $\mathbf{S}$ be a skew-symmetric matrix, i.e., $\mathbf{S}=-\mathbf{S}^H$.

(a) Prove that the diagonal entries of a complex skew-symmetric matrix are purely imaginary. (2 points)

Solution: Consider

$$\begin{aligned} \mathbf{S} &= -\mathbf{S}^H \\ \mathbf{S} + \mathbf{S}^H &= 0 \end{aligned}$$

Consider i^{th} diagonal element of $\mathbf{S} + \mathbf{S}^H$,

$$\begin{aligned} [\mathbf{S} + \mathbf{S}^H]_{ii} &= \mathbf{S}_{ii} + \mathbf{S}_{ii}^* = 0 \\ &\implies \text{Re}(\mathbf{S}_{ii}) = 0. \end{aligned}$$

Therefore the diagonal entries are purely imaginary.

(b) Prove that the eigenvalues of a real skew-symmetric matrix are either 0 or purely imaginary. (4 points)

Solution: We start with the eigenvalue-eigenvector equation:

$$\begin{aligned} \mathbf{S}\mathbf{x} &= \lambda \mathbf{x} \\ \mathbf{x}^H \mathbf{S}\mathbf{x} &= \lambda \mathbf{x}^H \mathbf{x} \\ &= \lambda \|\mathbf{x}\|^2 \end{aligned}$$

Also, we have that

$$\begin{aligned} \mathbf{x}^H(\mathbf{S}\mathbf{x}) &= ((\mathbf{S}\mathbf{x})^H \mathbf{x})^H \\ &= (\mathbf{x}^H \mathbf{S}^{\mathbf{x}})^H \\ &= (-\mathbf{x}^H \mathbf{S}\mathbf{x})^H \\ \mathbf{x}^H(\mathbf{S}\mathbf{x}) &= (-\mathbf{x}^H \mathbf{S}\mathbf{x})^H \implies \text{Re}(\mathbf{x}^H \mathbf{S}\mathbf{x}) = 0 \end{aligned}$$

Putting the above together,

$$\begin{aligned} \text{Re}(\mathbf{x}^H \mathbf{S}\mathbf{x}) &= \|\mathbf{x}\|^2 \text{Re}(\lambda) = 0 \\ \implies \lambda &= 0 \text{ or purely imaginary.} \end{aligned}$$

(c) If $\mathbf{S}$ is real-valued, can $(\mathbf{I} - \mathbf{S})$ be singular? Justify your answer. (6 points)

Solution:
Evals of $\mathbf{I} - \mathbf{S}$ are $1-$ (Evals of $\mathbf{S}$), and $\mathbf{S}$ has zero or purely imaginary eigenvalues. Therefore, the real part of the eigenvalues of $\mathbf{I} - \mathbf{S}$ is always 1, i.e., the eigenvalues can never be zero. Thus, determinant which is product of eigenvalues cannot be zero. Hence, $\mathbf{I} - \mathbf{S}$ is always non-singular.

4. **(Inner products)** For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, consider the usual inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ and the vector norm $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$.

(a) If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ are a set of orthonormal vectors in $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, show that

$$\sum_{i=1}^{r} |\langle \mathbf{x}, \mathbf{x}_i \rangle|^2 \leqslant \|\mathbf{x}\|^2.$$

When will equality be attained? (8 points)

Solution: Consider

$$\begin{aligned}
0 &\leqslant \|\mathbf{x} - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i\|^2 \\
&= \left(\mathbf{x} - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i\right)^T \left(\mathbf{x} - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i\right) \\
&= \|\mathbf{x}\|^2 - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}^T \mathbf{x}_i - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i^T \mathbf{x} + \sum_{i,j=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \langle \mathbf{x}, \mathbf{x}_j \rangle \mathbf{x}_i^T \mathbf{x}_j
\end{aligned}$$

Since $\{\mathbf{x}_i\}$ are a set of orthonormal vectors, $\sum_{i,j=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \langle \mathbf{x}, \mathbf{x}_j \rangle \mathbf{x}_i^T \mathbf{x}_j = \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle^2$, which is the same as the (negative of) the second and third terms. Thus,

$$\|\mathbf{x}\|^2 - \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle^2 \geqslant 0,$$

and the result is proved. Since $\|\mathbf{y}\| = 0$ if and only if $\mathbf{y} = 0$, equality is attained when

$$\mathbf{x} = \sum_{i=1}^{r} \langle \mathbf{x}, \mathbf{x}_i \rangle \mathbf{x}_i.$$

In other words, equality is attained when $\mathbf{x}$ lies in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$.

(b) Show that $\langle \mathbf{x}, \mathbf{y} \rangle \leqslant (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)/2$. (6 points)

Solution: Consider $\|\mathbf{x} - \mathbf{y}\|^2 = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle$. We have

$$\begin{aligned}
0 &\leqslant \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \\
&= (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \\
&= \mathbf{x}^T \mathbf{x} + \mathbf{y}^T \mathbf{y} - \mathbf{y}^T \mathbf{x} - \mathbf{x}^T \mathbf{y} \\
&= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle
\end{aligned}$$

The above implies that

$$\langle \mathbf{x}, \mathbf{y} \rangle \leqslant \frac{1}{2} \left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)$$

as desired.

5. (**Rank 1 matrix norms**) Suppose $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$, and let $\mathbf{E} = \mathbf{u}\mathbf{v}^T$. Show that

(a) Frobenius norm: $\|\mathbf{E}\|_F = \|\mathbf{u}\|_2\|\mathbf{v}\|_2$ (2 points)

Solution:

$$\|\mathbf{E}\|_F = \sqrt{\operatorname{tr}\left(\mathbf{E}\mathbf{E}^\top\right)} = \sqrt{\operatorname{tr}\left(\mathbf{u}\mathbf{v}^\top\mathbf{v}\mathbf{u}^\top\right)} = \sqrt{\left(\mathbf{u}^\top\mathbf{u}\right)\left(\mathbf{v}^\top\mathbf{v}\right)} = \|\mathbf{u}\|_2\,\|\mathbf{v}\|_2$$

$$\|\mathbf{E}\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|u_i v_j|^2 = \sum_{i=1}^{m}|u_i|^2\sum_{j=1}^{n}|v_j|^2 = \|\mathbf{u}\|_2^2\,\|\mathbf{v}\|_2^2$$

(b) Spectral norm: $|||\mathbf{E}|||_2 = \|\mathbf{u}\|_2\|\mathbf{v}\|_2$ (2 points)

Solution:

$$\|\mathbf{E}\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{E}\mathbf{x}\|_2 = \max_{\|\mathbf{x}\|_2=1}\left\|\mathbf{u}\mathbf{v}^\top\mathbf{x}\right\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{u}\|_2\,|\mathbf{v}^\top\mathbf{x}| = \|\mathbf{u}\|_2\,\|\mathbf{v}\|_2$$

(c) Maximum row sum norm: $|||\mathbf{E}|||_\infty = \|\mathbf{u}\|_\infty\|\mathbf{v}\|_1$ (2 points)

Solution:

$$\begin{aligned}\|\mathbf{E}\|_\infty &= \max_{\|\mathbf{x}\|_\infty=1}\|\mathbf{E}\mathbf{x}\|_\infty = \max_{\|\mathbf{x}\|_\infty=1}\left\|\mathbf{u}\mathbf{v}^\top\mathbf{x}\right\|_\infty = \max_{\|\mathbf{x}\|_\infty=1}\|\mathbf{u}\|_\infty\,|\mathbf{v}^\top\mathbf{x}| \\ &= \|\mathbf{u}\|_\infty\max_{\|\mathbf{x}\|_\infty=1}|\mathbf{v}^\top\mathbf{x}| = \|\mathbf{u}\|_\infty\,\|\mathbf{v}\|_1\end{aligned}$$

where the last equality holds since it is the definition of dual norm of the ℓ_∞ norm which is the ℓ_1 norm.

6. A 4×4 matrix $\mathbf{B}$ has eigenvalues $0, 1, 3, 5$.

(a) What is the rank of $\mathbf{B}$? (2 points)

Solution:

$\mathbf{B}$ has 4 distinct eigenvalues $\implies$ $\mathbf{B}$ is diagonalizable, i.e., it is similar to diagonal matrix containing its eigenvalues on its diagonal. Since exactly one of the eigenvalues is 0, the diagonal matrix has rank = 3; and since the rank of a matrix is similarity-invariant, the matrix $\mathbf{B}$ also has rank 3.

(b) What is the determinant of $\mathbf{B}^T\mathbf{B}$? (2 points)

Solution:

$\det(\mathbf{B}^T\mathbf{B}) = |\det(\mathbf{B})|^2 = (0 \times 1 \times 3 \times 5)^2 = 0$

(c) Calculate the eigenvalues of $(\mathbf{B}^2 + 5\mathbf{I})^{-1}$ (2 points)

Solution:

Eigenvalues of $\mathbf{B}^2$ are squares of eigenvalues of $\mathbf{B}$ which are $0, 1, 9, 25$.

Eigenvalues of $\mathbf{B}^2 + 5\mathbf{I}$ are (5 + eigenvalues of $\mathbf{B}^2$) which are $5, 6, 14, 30$.

Eigenvalues of $(\mathbf{B}^2+5\mathbf{I})^{-1}$ are inverses of eigenvalues of $(\mathbf{B}^2+5\mathbf{I})$ which are $\frac{1}{5}, \frac{1}{6}, \frac{1}{14}$ and $\frac{1}{30}$.

(d) If another 4×4 matrix $\mathbf{C}$ has eigenvalues $1, 2, 3, 6$, find the eigenvalues of the 8×8 matrix (4 points)

$$\mathbf{D} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

where $\mathbf{I}$ is the 4×4 identity matrix and $\mathbf{0}$ is the 4×4 all zero matrix.

Solution: Let $\mathbf{D} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, and consider $\det(\mathbf{D} - \lambda\mathbf{I}) = 0$. In the expansion of the determinant, all the entries of $\mathbf{C}$ are multiplied by zero. Hence, it has no effect on the eigenvalues of $\mathbf{D}$. Therefore, the eigenvalues of $\mathbf{D}$ are eigenvalues of $\mathbf{B}$ and $\mathbf{I}$ which is $0, 1, 3, 5, 1, 1, 1, 1$.

Alternative solution: $\det(\mathbf{D} - \lambda\mathbf{I}) = \det(\mathbf{B} - \lambda_B\mathbf{I})\det(\mathbf{I} - \lambda_I\mathbf{I})$ (Since the matrix $\mathbf{D}$ is block upper triangular). Hence, the eigenvalues of $\mathbf{D}$ are eigenvalues of $\mathbf{B}$ and $\mathbf{I}$ put together.