# E9 203: Homework - 1

Assigned on: 17 Jan. 2020

## 1 Topics

- Review of norms and related concepts

## 2 Problems

1. What condition(s) must exist on $\mathbf{b}$ so that the system of equations $A\mathbf{x} = \mathbf{b}$ has an exact solution?

2. Show that the solution to
$$\min_{\mathbf{x}} J(\mathbf{x}) \triangleq \|B\mathbf{x}\|_2^2 \text{ s. t. } A\mathbf{x} = \mathbf{b},$$
where $A \in \mathbb{R}^{m\times N}, B \in \mathbb{R}^{N\times N}, \mathbf{x} \in \mathbb{R}^N, \mathbf{b} \in \mathbb{R}^m$ and when $B^TB$ is invertible, is given by
$$\hat{\mathbf{x}} = (B^TB)^{-1}A^T(A(B^TB)^{-1}A^T)^{-1}\mathbf{b}.$$

3. Assume $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $A = I + \mathbf{u}\mathbf{v}^T$. Show that if $A$ is nonsingular, then $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$, for some scalar $\alpha$. Find the corresponding $\alpha$.

4. (Golub and Van Loan, P2.2.2) Prove the Cauchy-Schwartz inequality
$$\left|\mathbf{x}^T\mathbf{y}\right| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2. \tag{1}$$
*Hint:* Use the inequality $(a\mathbf{x} + b\mathbf{y})^T(a\mathbf{x} + b\mathbf{y}) \geq 0$ for suitable scalars $a$ and $b$.

5. (Golub and Van Loan, P2.2.4) Show that, if $\mathbf{x} \in \mathbb{R}^n$,
$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2$$
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$$
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\|\mathbf{x}\|_\infty$$
When is the equality attained?

6. (Golub and Van Loan, P2.2.7) Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if $\text{rank}(A) = n$, then $\|\mathbf{x}\|_A \triangleq \|A\mathbf{x}\|$ is a vector norm on $\mathbb{R}^n$.

7. (Golub and Van Loan, P2.2.8) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(\alpha) \triangleq \|\mathbf{x} - \alpha\mathbf{y}\|_2$. Show that $\psi$ is minimized when $\alpha = \mathbf{x}^T\mathbf{y}/\mathbf{y}^T\mathbf{y}$.

8. (A first problem involving sparsity) We are given 12 coins, and we are told that one of them is defective, i.e., that it is either heavier or lighter than the others. We are given a weighing balance (with no weights), i.e., it can be used to compare weights of groups of coins. Show that one can correctly identify the defective coin, and figure out whether it is heavier or lighter than the non-defective ones, in just 3 weighings.

   *Bonus question:* Generalize to $n$ coins, one of which is defective.