E2 212 Test 2 Solutions

1. (6 points) Let A and B be real symmetric matrices with A positive definite. Let

$$\rho_{A,B}(\mathbf{v}) = \frac{\mathbf{v}^H B \mathbf{v}}{\mathbf{v}^H A \mathbf{v}}.$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the roots of the polynomial $\det(B - \lambda A)$. Show that $\lambda_1 \leq \rho_{A,B}(\mathbf{v}) \leq \lambda_n$ for all $\mathbf{v} \neq 0$.

Solution: Since A is positive definite, we know that the roots of $\det(B - \lambda A) = 0$ correspond to the eigenvalues of the matrix $A^{-1}B$, because $\det(B - \lambda A) = \det(A)\det(A^{-1}B - \lambda I)$. Note that, using the Cholesky factorization, A can be written as $A = D^H D$. Define $C \triangleq D^{-H} B D^{-1}$ and $\mathbf{y} \triangleq D\mathbf{v}$. Then,

$$\rho_{A,B}(\mathbf{v}) = \frac{\mathbf{v}^H D^H D^{-H} B D^{-1} D \mathbf{v}}{\mathbf{v}^H D^H D \mathbf{v}} = \frac{\mathbf{y}^H C \mathbf{y}}{\mathbf{y}^H \mathbf{y}} \leq \max_{\mathbf{y}:\mathbf{y}^H\mathbf{y}=1} \mathbf{y}^H C \mathbf{y} = \lambda_{max}(C).$$

Similarly,

$$\rho_{A,B}(\mathbf{v}) = \frac{\mathbf{y}^H C \mathbf{y}}{\mathbf{y}^H \mathbf{y}} \geq \min_{\mathbf{y}:\mathbf{y}^H\mathbf{y}=1} \mathbf{y}^H C \mathbf{y} = \lambda_{min}(C).$$

Note that the $\mathbf{y}$ that maximizes/minimizes the right hand side is the eigenvector of C corresponding to its maximum/minimum eigenvalue. Moreover, the eigenvalues of C and the matrix $A^{-1}B$ are the same. This is because $C\mathbf{y} = \lambda\mathbf{y}$ implies $D^{-H}BD^{-1}\mathbf{y} = D^{-H}BD^{-1}D\mathbf{v} = D^{-H}B\mathbf{v} = \lambda D\mathbf{v}$. Premultiplying by D^{-1} and recognizing that λ commutes with D^{-1} as it is a scalar, we get $D^{-1}D^{-H}B\mathbf{v} = A^{-1}B\mathbf{v} = \lambda D^{-1}D\mathbf{v} = \lambda\mathbf{v}$. Thus, $C\mathbf{y} = \lambda\mathbf{y}$ implies $A^{-1}B\mathbf{v} = \lambda\mathbf{v}$, i.e., there is a 1-1 correspondence between eigenvectors of C and $A^{-1}B$, and they have the same eigenvalues. Hence, the result follows.

2. (6 points) Suppose we partition $A \in \mathbf{R}^{n\times n}$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbf{R}^{r\times r}$ is non-singular. Then $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the *Schur complement* of A_{11} in A. Show that, after r steps of the Gaussian elimination algorithm without pivoting, A_{22} gets overwritten by S.

Solution: Since Gaussian elimination is linear transform, it can be expressed as a matrix operator. Multiply the matrix operators for the first r steps and let G_r denote the transformation matrix (after the first r steps). Then, the Gaussian elimination implies

$$G_r A = A^{(r)},$$

where $A^{(r)}$ is such that the top-left $r \times r$ block is upper triangular. When written in partitioned form,

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{(r)} & A_{12}^{(r)} \\ 0 & A_{22}^{(r)} \end{bmatrix},$$

where $A_{11}^{(r)}$ is upper triangular. Equating the zero block on the right hand side above with the left hand side, we have

$$G_{21}A_{11} + G_{22}A_{21} = 0,$$

which implies

$$G_{21} = -G_{22}A_{21}A_{11}^{-1}$$

Now the Gaussian elimination process results in a lower triangular G matrix, and in particular, we have $G_{12} = 0$ and $G_{22} = I$. Thus, we get,

$$A_{22}^{(r)} = G_{21}A_{12} + G_{22}A_{22} = -A_{21}A_{11}^{-1}A_{12} + A_{22}.$$

3. (6 points) Suppose $\mathbf{x} = A^{-1}b$. Show that if $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ (the error) and $\mathbf{r} = \mathbf{b} - A\hat{\mathbf{x}}$, then

$$\frac{\|\mathbf{r}\|}{\|A\|} \leq \|\mathbf{e}\| \leq \|A^{-1}\|\|\mathbf{r}\|.$$

Solution:

$$\begin{aligned} \|\mathbf{r}\| &= \|\mathbf{b} - A\hat{\mathbf{x}}\| = \|A\mathbf{x} - A\hat{\mathbf{x}}\| = \|A\mathbf{e}\| \\ &\leq \|A\|\|\mathbf{e}\| \\ \Rightarrow \frac{\|\mathbf{r}\|}{\|A\|} &\leq \|\mathbf{e}\|. \end{aligned}$$

$$\begin{aligned} \|\mathbf{e}\| &= \|\mathbf{x} - \hat{\mathbf{x}}\| = \|A^{-1}\mathbf{b} - \hat{\mathbf{x}}\| = \|A^{-1}\mathbf{r}\| \\ &\leq \|A^{-1}\|\|\mathbf{r}\| \\ \Rightarrow \|\mathbf{e}\| &\leq \|A^{-1}\|\|\mathbf{r}\|. \end{aligned}$$

4. (7 points) Given,

$$\begin{aligned} A\mathbf{x} &= \mathbf{b}, A \in \mathbb{R}^{n\times n}, 0 \neq \mathbf{b} \in \mathbb{R}^n \\ (A + \Delta A)\mathbf{y} &= \mathbf{b} + \Delta\mathbf{b}, \Delta A \in \mathbb{R}^{n\times n}, \Delta\mathbf{b} \in \mathbb{R}^n \end{aligned}$$

with $\|\Delta A\| \leq \epsilon\|A\|$ and $\|\Delta \mathbf{b}\| \leq \epsilon\|\mathbf{b}\|$. If $\epsilon\kappa(A) = r < 1$ then show that $A + \Delta A$ is non-singular and

$$\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} \leq \frac{1+r}{1-r}.$$

Hint: If $F \in \mathbb{R}^{n\times n}$ and $\|F\| < 1$ then $I - F$ is non-singular and

$$\|(I-F)^{-1}\| \leq \frac{1}{1-\|F\|}.$$

Solution: Since A is non singular, $(A + \Delta A) = A(I - F)$ where $F = -A^{-1}\Delta A$. And $\|F\| = \|A^{-1}\Delta A\| \leq \|A^{-1}\|\|\Delta A\| \leq \epsilon\|A^{-1}\|\|A\| = r < 1$. Hence $A + \Delta A$ is non-singular. Also,

$$\begin{aligned}
\mathbf{y} &= (A+\Delta A)^{-1}(\mathbf{b}+\Delta\mathbf{b}) \\
&= (A(I+A^{-1}\Delta A))^{-1}(\mathbf{b}+\Delta\mathbf{b}) \\
&= (I+A^{-1}\Delta A)^{-1}A^{-1}(\mathbf{b}+\Delta\mathbf{b}) \\
&= (I+A^{-1}\Delta A)^{-1}(A^{-1}\mathbf{b}+A^{-1}\Delta\mathbf{b}) \\
&= (I+A^{-1}\Delta A)^{-1}(\mathbf{x}+A^{-1}\Delta\mathbf{b})
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathbf{y}\| &\leq \|(I+A^{-1}\Delta A)^{-1})\|\|(\mathbf{x}+A^{-1}\Delta\mathbf{b})\| \\
&\leq \frac{1}{1-r}(\|\mathbf{x}\|+\|A^{-1}\|\|\Delta\mathbf{b}\|) \\
&\leq \frac{1}{1-r}(\|\mathbf{x}\|+\epsilon\|A^{-1}\|\|\mathbf{b}\|)
\end{aligned}$$

Since $\|\mathbf{b}\| = \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$

$$\begin{aligned}
\|\mathbf{y}\| &\leq \frac{1}{1-r}(\|\mathbf{x}\|+\epsilon\|A^{-1}\|\,\|A\|\,\|\mathbf{x}\|) \\
&= \frac{1}{1-r}(\|\mathbf{x}\|+r\|\mathbf{x}\|) = \left(\frac{1+r}{1-r}\right)\|x\|.
\end{aligned}$$

Hence the result follows.