E2-212 Matrix Theory: Final Exam

05 Dec. 2013

Rules

1. Please **write your name and roll number** on the first page of your answer book.
2. This exam counts for 50% of your final grade.
3. It is a closed book exam. There is no need for a calculator as the numerical calculations, if any, will be simple. Please leave your cell-phones turned off and in your bag.
4. Please indicate your final answer clearly (for example, by drawing a box around it) and provide justifications. Avoid mixing up answers to different questions, making it hard to evaluate.
5. The points assigned to a problem does not necessarily indicate its difficulty level. Generally, more "basic" questions carry more points.
6. **Code of conduct:** Cheating is, needless to say, strictly forbidden and will be taken extremely seriously, so don't even think about it. I reserve the right to immediately terminate your exam at any point in time if I suspect you of cheating. So, it is your responsibility to conduct yourself in a manner that is completely above board and beyond suspicion.

Problems

1. (4 points)(Norms) For $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $z \in \mathbb{C}$, the matrix $\mathbf{R}(z) \triangleq (z\mathbf{I} - \mathbf{A})^{-1}$ is called the *resolvant* of the matrix $\mathbf{A}$; it is used in spectral estimation. Prove that, if $|z| > |||\mathbf{A}|||$ for any induced matrix norm $|||\cdot|||$, then
$$|||\mathbf{R}(z)||| \leqslant \frac{1}{|z| - |||\mathbf{A}|||}.$$
2. (Congruence)
   (a) (3 points) Show that any Hermitian symmetric matrix $\mathbf{A}$ is congruent to a real diagonal matrix.
   (b) (3 points) Recall that a matrix $\mathbf{A}$ is said to be *positive definite* if $\mathbf{x}^H\mathbf{A}\mathbf{x} > 0$ for all vectors $\mathbf{x} \neq \mathbf{0}$. If $\mathbf{A}$ is a real symmetric positive definite matrix, show the there exists a real symmetric positive definite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.
   (c) (2 points) (*Polar decomposition*) If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is nonsingular, show that the symmetric matrix $\mathbf{A}\mathbf{A}^T$ is positive definite. Let $\mathbf{B}$ be a positive definite square root of $\mathbf{A}\mathbf{A}^T$ as in part (b). Show that $\mathbf{T} \triangleq \mathbf{B}^{-1}\mathbf{A}$ is orthonormal, and hence that $\mathbf{A} = \mathbf{B}\mathbf{T}$, where $\mathbf{T}$ is orthonormal and $\mathbf{B}$ is positive definite.

3. (Generalized Eigenvalues)

   (a) (2 points) Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$. Let $\mathbf{A}$ be non-singular. Show that the λ satisfying $\mathbf{Bx} = \lambda\mathbf{Ax}$ are the same as the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$.

   (b) (3 points) Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be Hermitian. Let $\mathbf{A}$ be positive definite. Let

$$\rho_{\mathbf{A},\mathbf{B}}(\mathbf{x}) = \frac{\mathbf{x}^H\mathbf{Bx}}{\mathbf{x}^H\mathbf{Ax}}.$$

   Let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ be the roots of the polynomial $\det(\mathbf{B}-\lambda\mathbf{A})$. Show that $\lambda_1 \leqslant \rho_{\mathbf{A},\mathbf{B}}(\mathbf{x}) \leqslant \lambda_n$ for all $\mathbf{x} \neq 0$.

4. (3 points) For $\mathbf{A} \in \mathbb{C}^{n\times n}$, prove that $\mathbf{x}^H\mathbf{Ax} = 0$ for all $\mathbf{x} \in \mathbb{C}^{n\times 1}$ implies $\mathbf{A} = 0$. Does $\mathbf{x}^T\mathbf{Ax} = 0$ for all $\mathbf{x} \in \mathbb{R}^{n\times 1}$ imply $\mathbf{A} = 0$? Why/why not?

5. Consider the matrix

$$\mathbf{A}_\epsilon = \begin{bmatrix} \frac{1}{1+\epsilon} & \frac{1}{1+\epsilon} \\ \frac{\epsilon^2}{1+\epsilon} & \frac{1}{1+\epsilon} \end{bmatrix}, \quad \epsilon > 0.$$

   (a) (2 points) Show that $\lambda_2 = 1$ is a simple eigenvalue of $\mathbf{A}_\epsilon$, that $\rho(\mathbf{A}_\epsilon) = \lambda_2 = 1$, and $|\lambda_1| < 1$.

   (b) (2 points) Show that

$$\mathbf{x} = \frac{1}{1+\epsilon}\begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \text{ and } \mathbf{y} = \frac{1+\epsilon}{2\epsilon}\begin{bmatrix} \epsilon \\ 1 \end{bmatrix}$$

   are eigenvectors of $\mathbf{A}_\epsilon$ and $\mathbf{A}_\epsilon^T$, respectively, corresponding to the eigenvalue $\lambda = 1$.

   (c) (3 points) Calculate $\mathbf{A}_\epsilon^m$ explicitly, $m = 1, 2, \ldots$.

   (d) (3 points) Show that

$$\lim_{m\to\infty} \mathbf{A}_\epsilon^m = \frac{1}{2}\begin{bmatrix} 1 & \epsilon^{-1} \\ \epsilon & 1 \end{bmatrix}.$$

   (e) (2 points) Calculate $\mathbf{xy}^T$ and comment.

   (f) (2 points) What happens if $\epsilon \to 0$?

6. (6 points) (Non-negative matrices) Establish the Collatz-Wielandt formula that says that the Perron root (spectral radius) for a $\mathbf{A} > 0$ is given by $r = \min_{\mathbf{x}\in\mathcal{P}} \quad g(\mathbf{x})$, where

$$g(\mathbf{x}) = \max_{1\leqslant i\leqslant n} \frac{[\mathbf{Ax}]_i}{x_i} \quad \text{and} \quad \mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} > 0\}.$$

7. (a) (6 points) Solve the following optimization problem:

$$\arg\min_{\mathbf{x}\in\mathbb{R}} \|\mathbf{Ax} - \mathbf{b}\|_p,$$

   where $\mathbf{A} = [1\ 1\ 1]^T$; $\mathbf{b} = [b_1\ b_2\ b_3]^T$, with real-valued $b_1 \geqslant b_2 \geqslant b_3 \geqslant 0$, for $p = 1, 2$ and ∞.

   (b) (4 points) Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, with $m \geqslant n$ and $\text{Rank}(\mathbf{A}) = n$, and given $\mathbf{b} \in \mathbb{R}^m$, solve

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_2.$$

   Does your solution require $\mathbf{A}$ to be full rank? Why/why not?