A Novel Sparse Bayesian Algorithm for Recovery of Spatially Correlated Sparse Vectors

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Agenda

Recap

Multiple Measurements

Multiple Signal Realizations

$$\mathbf{y}_i = \mathbf{A}_i \mathbf{x}_i + \mathbf{w}_i, \; i = \{1, 2, ...L\} \quad (1)$$

where,

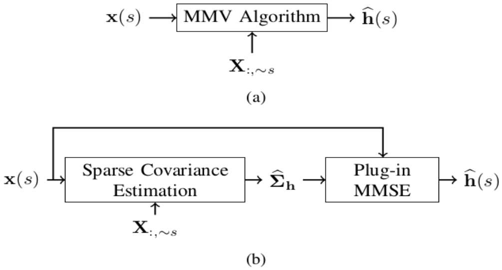
- $\mathbf{A}_i \in \mathbb{C}^{M \times N}, \forall i$ with $M \leq N$
- $\mathbf{w}_i \overset{iid}{\sim} \mathcal{CN}(\mathbf{0}, \sigma_n^2 \boldsymbol{I}) \quad \forall i$
- $\mathbf{Y} = \mathbf{AX} + \mathbf{W}$

$$p(\mathbf{Y}|\mathbf{X}, \sigma^2) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{y}_l; \mathbf{A}\mathbf{x}_l, \sigma_n^2 \mathbf{I}_M).$$

- Signal design: $\mathbf{x}_i \overset{iid}{\sim} \mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma})$

Summary

- Decoupling property: Covariance Estimation and Signal Recovery
- Advantage in exploiting correlation
- Sample Covariance: Samples of $O(N)$ even with N measurements
- Exploit structure in Covariance matrix (Eg. Masked Sample covariance)



[1]

Figure. (a) Generic MMV algorithm, (b) Decomposition of MMV into Covariance Estimation and plug-in MMSE estimator

[1] Haghighatshoar et al, 2019

Prior design

Prior models

Generative model for $\mathbf{x}$ (Structure of Σ)

- No spatial correlation
  - Diagonal covariance matrix
  - Few non-zero diagonals: Sparse
- Spatial correlation (Intra-vector correlation)
  - Non-diagonal covariance matrix
  - Toeplitz, Compound symmetry, Autoregressive etc
  - $K \times K$ non-zero sub matrix: Sparse
  - Block sparse structure

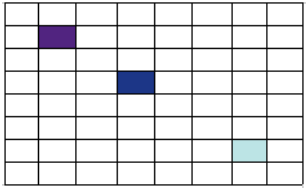


Figure. Diagonal Covariance

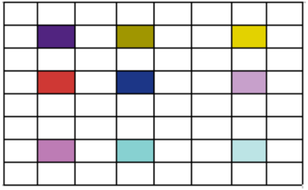


Figure. Covariance of sparse signals with spatial correlation

Correlation matrices

Correlation vs Covariance matrix

- Diagonal entries of correlation matrices are equal to 1
- Non-diagonal entries specify correlation coefficient
- Does not consider the effect of different variances of each index

Models for Correlation matrix **U**

- Compound Symmetry or Uniform correlation model
  - $U_{ij} = \begin{cases} 1 & \text{if } i = j \\ \rho & \text{if } i \neq j \end{cases}$
  - Diversity reception system for uniformly correlated Rayleigh fading channel [2]
- Auto-regressive or Exponential correlation model
  - $U_{ij} = \begin{cases} 1 & \text{if } i = j \\ \rho^{|i-j|} & \text{if } i \neq j \end{cases}$
  - mmWave Massive MIMO channel with correlated Rayleigh fading [3]

[2] Ranjan K Mallick, 2007
[3] Samimi, et.al., 2016

Building covariance matrices

Covariance model

- $p(\mathbf{x}|\boldsymbol{\gamma}, \mathbf{U}) = \mathcal{CN}(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\gamma}})$
- Hyper-parameter for variance $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1, & \gamma_2, & \dots & \gamma_n \end{bmatrix}^T$ with $\boldsymbol{\Gamma} = diag(\boldsymbol{\gamma})$
- $\rho_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y}$
- $(\boldsymbol{\Sigma}_{\boldsymbol{\gamma}})_{ij} = \sqrt{\gamma_i}\sqrt{\gamma_j}U_{ij}$

$$\Longrightarrow \boldsymbol{\Sigma}_{\boldsymbol{\gamma}} = \boldsymbol{\Gamma}^{\frac{1}{2}}\mathbf{U}\boldsymbol{\Gamma}^{\frac{1}{2}}$$

- λ as hyper-parameter for noise variance σ^2

Bayesian inference

Cost function

- Estimation of hyper-parameters $\theta = \{\boldsymbol{\gamma}, \lambda\}$
- Marginalized pdf: $p(\mathbf{y}|\theta) = \int_{\mathbf{x}} p(\mathbf{y}|\mathbf{x}, \lambda) p(\mathbf{x}|\gamma, \mathbf{U}) \, \mu d\mathbf{x}$

$$\Longrightarrow p(\mathbf{y}|\theta) = \frac{1}{(2\pi)^{M/2}} \frac{1}{|\boldsymbol{\Sigma}_{\mathbf{y}}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{y}^H \boldsymbol{\Sigma}_{\mathbf{y}}^{-1} \mathbf{y}\right)$$

where $\boldsymbol{\Sigma}_{\mathbf{y}} = \left[\frac{\mathbf{I} - \mathbf{A}\left[\lambda \boldsymbol{\Sigma}_{\gamma}^{-1} + \mathbf{A}^H \mathbf{A}\right]\mathbf{A}^H}{\lambda}\right]^{-1} = \lambda \mathbf{I} + \mathbf{A} \boldsymbol{\Sigma}_{\gamma} \mathbf{A}^H$

- Evidence maximization of θ: Maximizing $p(\mathbf{y}|\theta)$
- Alternatively, minimizing

$$L(\gamma, \mathbf{U}, \lambda) = -log(p(\mathbf{y}|\theta)) = \frac{1}{2}\log(|\boldsymbol{\Sigma}_{\mathbf{y}}|) + \frac{1}{2}\mathbf{y}^H \boldsymbol{\Sigma}_{\mathbf{y}}^{-1} \mathbf{y}$$

Expectation-Maximization

- Special case of Majorization Minimization (MM) algorithms
- Local maximum likelihood parameters
- Set of incomplete observed data (**y**) assumed to be generated from a specific model

Algorithm

1. Consider set of starting parameters.
2. Expectation-step: Estimate the variables using the observed data
3. Maximization-step: Use complete data ($\mathbf{x}, \mathbf{y}$) obtained in E-step to update parameters or hypothesis
4. Check whether values are converging or not

Analytic expression

$$\theta_{new} = \arg\max_{\theta} \mathbb{E}_{\mathbf{x}|\mathbf{y},\theta_{old}} [\log(p(\mathbf{x}, \mathbf{y}|\theta))]$$

E-Step

Posterior distribution

$$p(\mathbf{x}|\mathbf{y},\theta) = \mathcal{CN}(\mathbf{x};\boldsymbol{\mu}_\theta,\boldsymbol{\Sigma}_\theta)$$

where, $\boldsymbol{\Sigma}_\theta = \left(\frac{1}{\lambda}\mathbf{A}^H\mathbf{A} + \boldsymbol{\Sigma}_\gamma^{-1}\right)^{-1}$ and $\boldsymbol{\mu}_\theta = \frac{1}{\lambda}\boldsymbol{\Sigma}_\theta\mathbf{A}^H\mathbf{y}$.

Expectation step

$$Q(\theta_{old},\theta) = \mathbb{E}_{\mathbf{x}|\mathbf{y},\theta_{old}}\left[\log(p(\mathbf{x},\mathbf{y}|\theta))\right] = \mathbb{E}_{\mathbf{x}|\mathbf{y},\theta_{old}}\left[\log(p(\mathbf{x}|\gamma,\mathbf{U}))\right] + \mathbb{E}_{\mathbf{x}|\mathbf{y},\theta_{old}}\left[\log(p(\mathbf{y}|\mathbf{x},\lambda))\right]$$

$$= Q_1(\theta_{old},\gamma,\mathbf{U}) + Q_2(\theta_{old},\lambda)$$

where,

$$Q_1(\theta_{old},\gamma,\mathbf{U}) = c - \frac{1}{2}log(|\boldsymbol{\Sigma}_\gamma|) - \frac{1}{2}Tr\left[\boldsymbol{\Sigma}_\gamma^{-1}\boldsymbol{\Sigma}_{old}\right],$$

$$Q_2(\theta_{old},\lambda) = c - \frac{M}{2}\log(\lambda) - \frac{\|\mathbf{y}\|^2}{2\lambda} + \frac{\boldsymbol{\mu}_\theta^H\mathbf{A}^H\mathbf{y}}{\lambda} - \frac{1}{2\lambda}Tr\left[\mathbf{A}^H\mathbf{A}\boldsymbol{\Sigma}_{old}\right],$$

where $\boldsymbol{\Sigma}_{old} = \boldsymbol{\Sigma}_{\theta_{old}} + \boldsymbol{\mu}_{\theta_{old}}\boldsymbol{\mu}_{\theta_{old}}^H$.

Optimality condition

Gamma update

$$\boldsymbol{\gamma}^* = \arg\max_{\boldsymbol{\gamma}} Q_1(\theta_{old}, \boldsymbol{\gamma})$$

Optimality condition

$$\begin{aligned}
0 =& -\frac{1}{2}\frac{\partial\left(\log|\boldsymbol{\Gamma}| + \log|\mathbf{U}| + Tr\left[\left(\boldsymbol{\Gamma}^{-\frac{1}{2}}\mathbf{P}\boldsymbol{\Gamma}^{-\frac{1}{2}}\right)\boldsymbol{\Sigma}_{old}\right]\right)}{\partial\gamma_i} \\
=& \frac{1}{\gamma_i} - Tr\left[\left(\frac{1}{2\gamma_i^{\frac{3}{2}}}\mathbf{J}_{ii}\mathbf{P}\boldsymbol{\Gamma}^{-\frac{1}{2}} + \frac{1}{2\gamma_i^{\frac{3}{2}}}\boldsymbol{\Gamma}^{-\frac{1}{2}}\mathbf{P}\mathbf{J}_{ii}\right)\boldsymbol{\Sigma}_{old}\right] \text{ where } \mathbf{J}_{ii} \text{ is single entry matrix} \\
=& 2\sqrt{\gamma_i} - 2\,Tr\left[\mathbf{J}_{ii}\mathbf{P}\boldsymbol{\Gamma}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{old}\right] \\
=& 2\sqrt{\gamma_i} - 2\left(\mathbf{P}\boldsymbol{\Gamma}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{old}\right)_{ii}.
\end{aligned}$$

$$\Rightarrow \sqrt{\gamma_i} = \sum_{k=1}^{N}\frac{1}{\sqrt{\gamma_k}}P_{ik}(\Sigma_{old})_{ki}\ \forall i \in [N].$$

Fixed point iterations

- Optimality condition has variables dependent on each other
- Let $c_i = 1/\sqrt{\gamma_i}$

$$\frac{1}{\mathbf{c}} = \mathbf{K}\mathbf{c},$$

$$\Rightarrow \mathbf{c} = \mathbf{K}^{-1}\frac{1}{\mathbf{c}}$$

where $\frac{1}{\mathbf{c}}$ is element wise inverse of $\mathbf{c}$ and $\mathbf{K} = \mathbf{P} \odot \boldsymbol{\Sigma}_{old}$.

- One iteration leads to reduction in cost function for each EM iteration
- $\mathbf{c}_{new} = \mathbf{K}^{-1}\frac{1}{\mathbf{c}_{old}}$ and update $\gamma_i = \left(\frac{1}{c_i}\right)^2, \qquad \forall i \in [N].$

Noise variance update

$$\lambda^* = \arg\max_{\lambda} Q_2(\theta_{old}, \lambda).$$

Optimality condition

$$0 = \frac{\partial Q_2}{\partial \lambda}$$

$$= \frac{\partial \left(c - \frac{M}{2}\log(\lambda) - \frac{\mathbf{y}^H\mathbf{y}}{2\lambda} + \frac{\boldsymbol{\mu}_{old}^H \mathbf{A}^H \mathbf{y}}{\lambda} - \frac{Tr\left(\mathbf{A}^H\mathbf{A}\boldsymbol{\Sigma}_{old}\right)}{2\lambda} \right)}{\partial \lambda}$$

$$= \frac{-1}{2\lambda^*} M + \frac{1}{2{\lambda^*}^2} \left[\mathbf{y}^H\mathbf{y} - 2\boldsymbol{\mu}_{old}^H \mathbf{A}^H\mathbf{y} + Tr\left(\mathbf{A}^H\mathbf{A}\boldsymbol{\Sigma}_{old}\right)\right]$$

$$\Rightarrow \lambda^* = \frac{1}{M}\left[\mathbf{y}^H\mathbf{y} - 2\mu_{old}^H\mathbf{A}^H\mathbf{y} + Tr\left[\mathbf{A}^H\mathbf{A}\boldsymbol{\Sigma}_{old}\right]\right]$$

Algorithm

Algorithm 1: Correlated EM

Input: $\mathbf{A}, \mathbf{Y}, \mathbf{U}$

Initialize: $k \leftarrow 0, \boldsymbol{\gamma} \leftarrow \mathbf{1}, \lambda \leftarrow 0.1$

while $k \leq k_{max}$ *and* $\boldsymbol{\mu}_\theta \neq \boldsymbol{\mu}_{\theta_{old}}$ **do**

$$\boldsymbol{\Sigma}_{\theta_{old}} = \left(\frac{1}{\lambda}\mathbf{A}^H\mathbf{A} + \boldsymbol{\Sigma}_{\boldsymbol{\gamma}}^{-1}\right)^{-1}$$

$$\boldsymbol{\mu}_{\theta_{old}} = \frac{1}{\lambda}\boldsymbol{\Sigma}_{\theta_{old}}\mathbf{A}^H\mathbf{Y}$$

$$\boldsymbol{\Sigma}_{old} = \boldsymbol{\Sigma}_{\theta_{old}} + \boldsymbol{\mu}_{\theta_{old}}\boldsymbol{\mu}_{\theta_{old}}^H$$

$$c_i = \sqrt{\frac{1}{\gamma_i}}\ \forall i$$

$$\mathbf{c} \leftarrow \left(\mathbf{U}^{-1} \odot \boldsymbol{\Sigma}_{old}\right)^{-1}\frac{1}{\mathbf{c}}$$

$$\gamma_i = \left(\frac{1}{c_i}\right)^2$$

$$\lambda^* = \frac{1}{M}\left[\frac{1}{L}\|\mathbf{Y}\|_F^2 - \frac{2}{L}\boldsymbol{\mu}_{old}^H\mathbf{A}^H\mathbf{Y} + Tr\left[\mathbf{A}^H\mathbf{A}\boldsymbol{\Sigma}_{old}\right]\right]$$

$$k \leftarrow k + 1$$

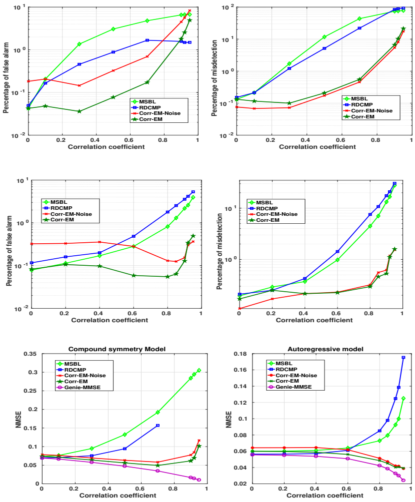
end

Output: $\mathbf{X} = \boldsymbol{\mu}_\theta, \boldsymbol{\gamma}$, Support obtained by non-zero entries of $\boldsymbol{\gamma}$

Results

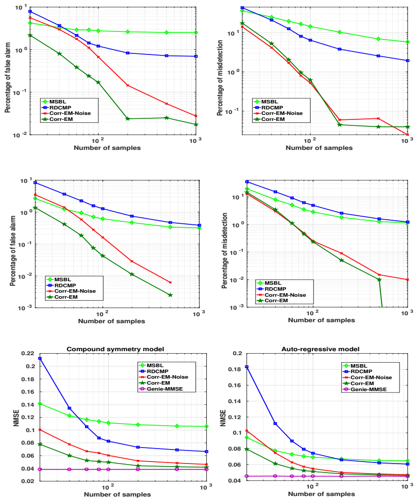
Effect of correlation

- $N = 100$
- $K = 20$
- $M = 20$
- $L = 100$
- Row1 : Support recovery for uniform model
- Row2 : Support recovery for exponential model
- Row3 : NMSE recovery for both models



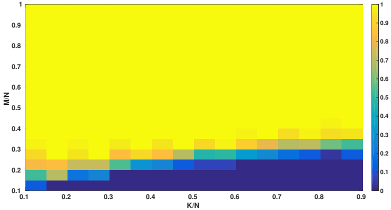
Effect of number of samples

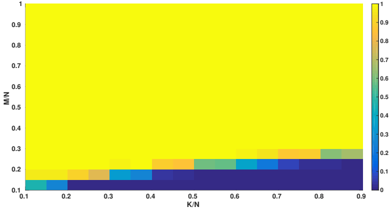
- $N = 100$
- $K = 20$
- $M = 20$
- Row1 : Support recovery for uniform model $\rho = 0.5$
- Row2 : Support recovery for exponential model $\rho = 0.75$
- Row3 : NMSE recovery for both models



Phase transition

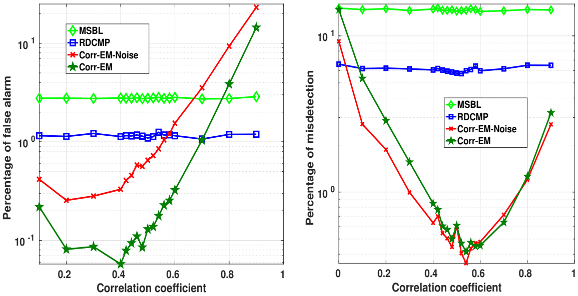
- $N = 100$
- $L = 100$
- Figure 1: Uniform model $\rho = 0.5$
- Figure 2 : Exponential model $\rho = 0.75$



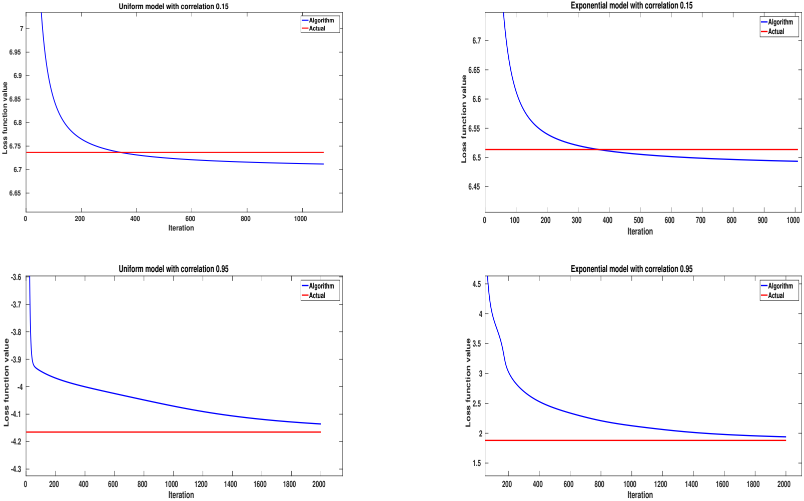


Error in correlation coefficient

- $N = 100$
- $K = 20$
- $M = 20$
- $L = 100$
- Uniform model
- Actual $\rho = 0.5$



Does the EM converge?



Challenges

Theory behind the algorithm?

Open Questions

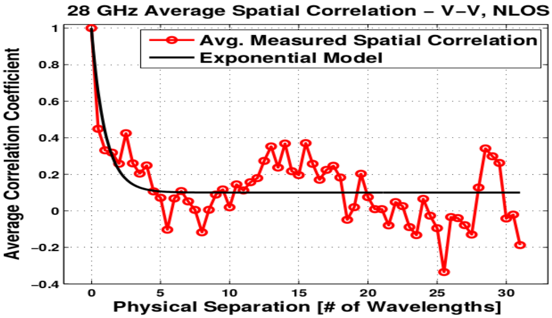
- Why only single iteration of fixed point is useful?
- Does the EM converge to a stationary point? What are the properties? How do you show convergence?
- What happens when exact correlation coefficient is not known?
- Can we learn the correlation coefficients from the available data?
- How useful is the algorithm for practical scenarios?
- What is the computational complexity? Can we make it faster?

Learning correlation coefficients

- Works only for applications with known **U**
- Parametric form for **U**
- Can we learn the parameters from sample covariance of $\hat{\mathbf{x}} = \mathbf{A}^T\mathbf{y}$?
- Simplify the optimality conditions for **U** or its parameters as hyper-parameters

Where can this be applied?

- Correlated Rayleigh fading in Massive MIMO channels
- Can models with correlation counter fundamental limit created due to pilot contamination? [4]



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[4] Sanguinetti, et.al., 2019
[5] Samimi, et.al., 2016

References

- M. Azizyan, A. Krishnamurthy, and A. Singh, “Extreme compressive sampling for covariance estimation,” *IEEE Transactions on Information Theory*, vol. 64, no. 12, pp. 7613–7635, Dec 2018.
- R. Y. Chen, A. Gittens, and J. A. Tropp, “The masked sample covariance estimator: an analysis using matrix concentration inequalities,” *Information and Inference: A Journal of the IMA*, vol. 1, no. 1, pp. 2–20, 2012.
- S. Haghighatshoar and G. Caire, “Multiple measurement vectors problem: A decoupling property and its applications,” 2018.
- L. Sanguinetti, E. Björnson, and J. Hoydis, “Towards massive mimo 2.0: Understanding spatial correlation, interference suppression, and pilot contamination,” *arXiv preprint arXiv:1904.03406*, 2019.
- R. K. Mallik, “The uniform correlation matrix and its application to diversity,” *IEEE Transactions on Wireless Communications*, vol. 6, no. 5, pp. 1619–1625, May 2007.
- M. K. Samimi, S. Sun, and T. S. Rappaport, “Mimo channel modeling and capacity analysis for 5g millimeter-wave wireless systems,” in *2016 10th European Conference on Antennas and Propagation (EuCAP)*, April 2016, pp. 1–5.
- L. Sanguinetti, E. Björnson, and J. Hoydis, “Towards massive mimo 2.0: Understanding spatial correlation, interference suppression, and pilot contamination,” 2019.