Block sparse Bayesian learning for extended source localization

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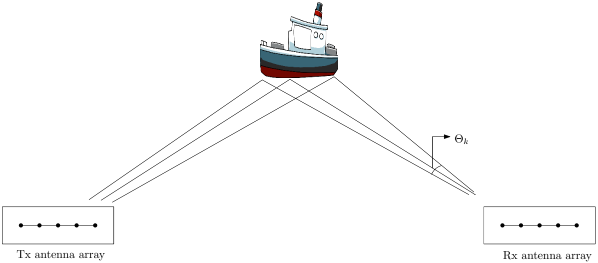
Introduction

- High resolution array processing - radar, radio astronomy, radio communications.
- Far-field point source assumption is an approximation.
- Distributed source- subspace based techniques fail.



System Model

- M_t Tx antennas, spacing Δ_t;
  M_r Rx antennas, spacing Δ_r.
- N_d Doppler bins, N_r range bins, N_a angular bins.
- $\mathbf{s}_i \in \mathbb{C}^{L\times 1}$: waveform transmitted by the ith Tx antenna.
- For the dth Doppler bin
  $\mathbf{s}_i(\omega_d) = \mathbf{s}_i \odot [1, e^{j\omega_d}, \ldots, e^{j(L-1)\omega_d}]^T$,
  $\mathbf{S}_d = [\mathbf{s}_1(\omega_d)\ \mathbf{s}_2(\omega_d)\ \cdots\ \mathbf{s}_{M_t}(\omega_d)]^T$.

System Model

- $\alpha_{d,r}^{(k)}(\theta)$: complex angular weighting function of the kth source in direction θ.
- $\mathbf{a}(\theta)$: Tx steering vector, $\mathbf{b}(\theta)$: Rx steering vector.
- Received signal:
$\mathbf{Y} = \sum_{d=1}^{N_d} \sum_{r=1}^{N_r} \sum_{k=1}^{K} \int_{\theta \in \Theta_k} \{\alpha_{d,r}^{(k)}(\theta)\mathbf{b}(\theta)\mathbf{a}^T(\theta)d\theta\}\tilde{\mathbf{S}}_d \mathbf{J}_r + \mathbf{W}$,
- $\tilde{\mathbf{S}}_d = [\mathbf{S}_d \quad \mathbf{0}_{M_t \times N_r - 1}]$, $\mathbf{J}_r = \begin{pmatrix} \overbrace{0 \ldots 01}^{r} & & \mathbf{0} \\ & \ddots & \\ & & 1 \\ \mathbf{0} & & \end{pmatrix}$.

System Model

- $\mathbf{Y} = \sum_{d=1}^{N_d} \sum_{r=1}^{N_r} \sum_{a=1}^{N_a} \alpha_{d,r,a} \mathbf{b}(\theta_a) \mathbf{a}^T(\theta_a) \tilde{\mathbf{S}}_d \mathbf{J}_r + \mathbf{W}$.

- Vectorize to get: $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}$.

- $\mathbf{A} = [\mathbf{u}_{1,1,1}\ \mathbf{u}_{1,1,2} \cdots \mathbf{u}_{N_d,N_r,N_a}]$,
  $\mathbf{u}_{d,r,a} = \mathrm{vec}(\mathbf{b}(\theta_a)\mathbf{a}^T(\theta_a)\tilde{\mathbf{S}}_d \mathbf{J}_r)$,
  $\mathbf{x} = [\alpha_{1,1,1}, \alpha_{1,1,2}, \ldots, \alpha_{N_d,N_r,N_a}]^T$.

- $\mathbf{x}$ is block-sparse.

Block sparse Bayesian learning

- BSBL-EM [Z. Zhang, and B. D. Rao, 2013]: Exploits intra-block correlation.

$p(\mathbf{x}_i; \gamma_i, \mathbf{B}_i) \sim \mathcal{N}(\mathbf{0}, \gamma_i \mathbf{B}_i) \forall i$
$p(\mathbf{x}; \{\gamma_i, \mathbf{B}_i\}_i) \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_0)$, $\mathbf{\Sigma}_0 = \mathrm{diag}(\gamma_1 \mathbf{B}_1, \gamma_2 \mathbf{B}_2, \ldots, \gamma_g \mathbf{B}_g)$
$p(\mathbf{x}|\mathbf{y}; \{\gamma_i, \mathbf{B}_i\}_i) \sim \mathcal{N}(\boldsymbol{\mu}_x, \mathbf{\Sigma}_x)$,
$\boldsymbol{\mu}_x = \mathbf{\Sigma}_0 \mathbf{A}^T (\lambda \mathbf{I} + \mathbf{A} \mathbf{\Sigma}_0 \mathbf{A}^T)^{-1} \mathbf{y}$,
$\mathbf{\Sigma}_x = (\mathbf{\Sigma}_0^{-1} + \frac{1}{\lambda} \mathbf{A}^T \mathbf{A})$

E-step: $\hat{\mathbf{x}} = \boldsymbol{\mu}_x$,.
M-step: $\gamma_i = \mathrm{tr}[\mathbf{B}_i^{-1} (\mathbf{\Sigma}_x^i + \boldsymbol{\mu}_x^i (\boldsymbol{\mu}_x^i)^T)]$.

Block sparse Bayesian learning

- Pattern-coupled BSBL (PC-BSBL): $\boldsymbol{\alpha}$ vector of hyperparameters.

$$p(\mathbf{x}|\boldsymbol{\alpha}) \sim \prod_{i=1}^{N} p(x_i|\alpha_i, \alpha_{i+1}, \alpha_{i-1})$$

$$p(x_i|\alpha_i, \alpha_{i+1}, \alpha_{i-1}) = \mathcal{N}(0, (\alpha_i + \beta\alpha_{i+1} + \beta\alpha_{i-1})^{-1}),$$

$$p(\alpha_i) = \Gamma(a)^{-1} b^a \alpha_i^a e^{-b\alpha_i}.$$

Proposed Method - I

- Assume hyperparameters:
$\gamma_1 = \gamma_1^{'} + \beta_1 \gamma_2^{'}$
$\gamma_2 = \beta_{-1} \gamma_1^{'} + \gamma_2^{'} + \beta_1 \gamma_3^{'}$
$\vdots$
$\gamma_N = \gamma_N^{'} + \beta_{-1} \gamma_{N-1}^{'}$
$\mathbf{\Gamma} = \mathrm{diag}\{\gamma_1^{'} + \beta_1 \gamma_2^{'},\ \beta_{-1} \gamma_1^{'} + \gamma_2^{'} + \beta_1 \gamma_3^{'},\ \ldots,\ \gamma_N^{'} + \beta_{-1} \gamma_{N-1}^{'}\}$

- $p(\mathbf{x}|\mathbf{y}; \gamma^{'}, \beta) = \mathcal{N}(\boldsymbol{\mu}_x, \mathbf{\Sigma}_x)$
$\boldsymbol{\mu} = \frac{1}{\lambda} \mathbf{\Sigma} \mathbf{A}^T \mathbf{y}; \quad \mathbf{\Sigma} = (\frac{1}{\lambda} \mathbf{A}^T \mathbf{A} + \mathbf{\Gamma}^{-1})^{-1}$

Proposed Method - I

- E-step: $\hat{\mathbf{x}} = \boldsymbol{\mu}$.
- M-step:

$$\begin{aligned}\gamma_i^{'} &= \underset{\gamma_i^{'} \geq 0}{\arg\max}\, \mathbb{E}_{\mathbf{x}|\mathbf{y};\boldsymbol{\gamma}',\boldsymbol{\beta}}[p(\mathbf{y}, \mathbf{x}; \boldsymbol{\gamma}^{'}, \boldsymbol{\beta})] \\ &= \underset{\gamma_i^{'} \geq 0}{\arg\max}\, \mathbb{E}_{\mathbf{x}|\mathbf{y};\boldsymbol{\gamma}',\boldsymbol{\beta}}[p(\mathbf{x}; \boldsymbol{\gamma}^{'}, \boldsymbol{\beta})]\end{aligned}$$

- $\gamma_i^{'} = \arg\max_{\gamma_i^{'} \geq 0} \log \mathbb{E}_{\mathbf{x}|\mathbf{y};\boldsymbol{\gamma}',\boldsymbol{\beta}}[p(\mathbf{y}, \mathbf{x}; \boldsymbol{\gamma}^{'}, \boldsymbol{\beta})]$

Proposed Method - I

- $\frac{\partial}{\partial \gamma_i^{'}} \log \mathbb{E}_{\mathbf{x}|\mathbf{y};\boldsymbol{\gamma}^{'},\boldsymbol{\beta}}[p(\mathbf{y},\mathbf{x};\boldsymbol{\gamma}^{'},\boldsymbol{\beta})] = 0$

$$\frac{\beta_1}{\beta_{-1}\gamma_{i-2}^{'}+\gamma_{i-1}^{'}+\beta_1\gamma_i^{'}} + \frac{1}{\beta_{-1}\gamma_{i-1}^{'}+\gamma_i^{'}+\beta_1\gamma_{i+1}^{'}} + \frac{\beta_{-1}}{\beta_{-1}\gamma_i^{'}+\gamma_{i+1}^{'}+\beta_1\gamma_{i+2}^{'}}$$
$$= \frac{\beta_1(\Sigma_{i-1,i-1}+\mu_{i-1}^2)}{(\beta_{-1}\gamma_{i-2}^{'}+\gamma_{i-1}^{'}+\beta_1\gamma_i^{'})^2} + \frac{\Sigma_{i,i}+\mu_i^2}{(\beta_{-1}\gamma_{i-1}^{'}+\gamma_i^{'}+\beta_1\gamma_{i+1}^{'})^2} + \frac{\beta_{-1}(\Sigma_{i+1,i+1}+\mu_{i+1}^2)}{(\beta_{-1}\gamma_i^{'}+\gamma_{i+1}^{'}+\beta_1\gamma_{i+2}^{'})^2}$$

Proposed Method - I

Update for $\gamma_i^{'}$

- Let $v_i = \frac{1}{\beta_{-1}\gamma_{i-1}^{'}+\gamma_i^{'}+\beta_1\gamma_{i+1}^{'}}$.

$$\frac{1}{\beta_{-1}\gamma_{i-1}^{'} + \gamma_i^{'} + \beta_1\gamma_{i+1}^{'}}$$
$$= (\Sigma_{i,i} + \mu_i^2)v_i^2 + ((\Sigma_{i-1,i-1} + \mu_{i-1}^2)v_{i-1} - 1)\beta_1 v_{i-1}$$
$$+ ((\Sigma_{i+1,i+1} + \mu_{i+1}^2)v_{i+1} - 1)\beta_{-1} v_{i+1}.$$

- $\gamma_i^{'} = \frac{1}{(\Sigma_{i,i}+\mu_i^2)v_i^2+((\Sigma_{i-1,i-1}+\mu_{i-1}^2)v_{i-1}-1)\beta_1 v_{i-1}+((\Sigma_{i+1,i+1}+\mu_{i+1}^2)v_{i+1}-1)\beta_{-1} v_{i+1}}$
$-\beta_{-1}\gamma_{i-1}^{'} - \beta_1\gamma_{i+1}^{'}$.

Method - II

- Consider 3 equations containing γ_i,

$$\beta_{-1}\gamma_{i-1}^{'} + \gamma_{i}^{'} + \beta_1\gamma_{i+1}^{'} = \frac{1}{\mathbb{E}[x_i^2]v_i^2 + (\mathbb{E}[x_{i-1}^2]v_{i-1} - 1)\beta_1 v_{i-1} + (\mathbb{E}[x_{i+1}^2]v_{i+1} - 1)\beta_{-1} v_{i+1}}$$

$$\beta_{-1}\gamma_{i-2}^{'} + \gamma_{i-1}^{'} + \beta_1\gamma_{i}^{'} = \frac{1}{\mathbb{E}[x_{i-1}^2]v_{i-1}^2 + (\mathbb{E}[x_{i-2}^2]v_{i-2} - 1)\beta_1 v_{i-2} + (\mathbb{E}[x_{i}^2]v_{i} - 1)\beta_{-1} v_{i}}$$

$$\beta_{-1}\gamma_{i}^{'} + \gamma_{i+1}^{'} + \beta_1\gamma_{i+2}^{'} = \frac{1}{\mathbb{E}[x_{i+1}^2]v_{i+1}^2 + (\mathbb{E}[x_{i}^2]v_{i} - 1)\beta_1 v_{i} + (\mathbb{E}[x_{i+2}^2]v_{i+2} - 1)\beta_{-1} v_{i+2}}$$

Some results for Method - I

- $M_t = 5, M_r = 5, N_d = 11, N_r = 12, N_a = 11, \mathrm{SNR} = 20\mathrm{dB}.$