Construction of Unimodular Tight Frames Using Majorization-Minimization for Compressed Sensing

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Abstract

In this paper, we propose a method to construct uni-modular tight frames (UMTFs), which are tight frames with the additional constraint that every entry of the matrix has the same magnitude. UMTFs are useful in many applications, since multiplication of a UMTF by a vector can be implemented in polar coordinates using very low computational cost. Since normalized UMTFs are unit norm tight frames (UNTFs), and since a UNTF is a minimizer of the frame potential, we propose an algorithm to find UMTFs by minimizing the frame potential. We show that minimizing the frame potential is equivalent to minimizing the total coherence when the frame is unimodular. We use the majorization-minimization approach to propose a low complexity, iterative, fast-converging algorithm for minimizing the frame potential. We also extend our algorithm to the cases where the phase angles of the sensing matrix are required to belong to a given finite set of feasible angles, and to the case where the signal being sampled is sparse in an arbitrary, possibly non-canonical basis. We illustrate the utility of our proposed construction in the context of sparse signal recovery. Partial DFT matrices, obtained by randomly selected rows from the full DFT matrix, are UMTFs. However, they perform poorly when dealing with signals that admit a sparse representation in the wavelet, Fourier and discrete cosine transform domains. In such scenarios, we illustrate the superior performance of our construction compared to the partial DFT, complex Gaussian and Bernoulli random matrices through simulations. The proposed algorithm offers the same performance as the partial DFT matrix, and outperforms the complex Gaussian and Bernoulli random matrices, when the signal is sparse in the canonical basis.

Keywords: Unimodular tight frames, Frame potential, total-coherence,
Majorization-Minimization methods, Compressed sensing.

1. Introduction

Frames are overcomplete (or redundant) sets of vectors that serve to faithfully represent signals. In the finite dimensional setting, frames are spanning sets. Although they were introduced in 1952 by Duffin and Schaeffer [1], they primarily gained popularity in the 1990s due to their application in wavelets [2]. Frames offer the advantage of redundancy in signal representations and numerical stability of reconstruction, and therefore have been increasingly studied in the signal processing community in the recent decades [3, 4]. There are an abundance of applications of frame theory in pure and applied mathematics, engineering, medicine and even quantum communication [3].

Grassmannian tight frames, which are tight frames with low coherence properties, have been used in Grassmannian packings, spherical codes and graph theory [5]. Therefore, frame theory and its applications have gained a growing interest among mathematicians, computer scientists and engineers.

A family of vectors $\{\boldsymbol{\psi}_i\}_{i=1}^M$ in $\mathbb{C}^m$ is called a *frame* for $\mathbb{C}^m$, if there exist constants $0 < a \leq b < \infty$ such that

$$a \|\mathbf{x}\|^2 \leq \sum_{i=1}^{M} |\langle \mathbf{x}, \boldsymbol{\psi}_i \rangle|^2 \leq b \|\mathbf{x}\|^2, \forall \mathbf{x} \in \mathbb{C}^m, \tag{1}$$

where a, b are called the lower and upper frame bounds, respectively, $\langle \cdot, \cdot \rangle$ represents the inner product between two vectors, and $\|\cdot\|$ represents the ℓ_2 norm of a vector. The matrix[1] $\Psi = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_M] \in \mathbb{C}^{m \times M}$, with $\boldsymbol{\psi}_i$ as its columns, is known as the frame synthesis operator, and is equivalent to the frame itself. The optimal frame bounds a and b are the least and greatest eigenvalues of $\Psi\Psi^H$, where $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operation.

If $a = b$ in (1), the frame is called an $a-$tight frame, and if $a = b = 1$, it is a Parseval frame. Tight frames are useful in applications, as they provide Parseval-like decompositions:

$$\mathbf{x} = \frac{1}{a} \sum_{i=1}^{M} \langle \mathbf{x}, \boldsymbol{\psi}_i \rangle \boldsymbol{\psi}_i, \forall \mathbf{x} \in \mathbb{C}^m \tag{2}$$

even when $\boldsymbol{\psi}_i$'s are not linearly independent. The tightness condition of a frame Ψ implies that the rows of Ψ are orthogonal and have equal norm $\sqrt{a}$. An a-tight frame is said to be a unit norm tight frame if $\|\boldsymbol{\psi}_i\| = 1; \forall i = 1, \ldots, M$. Any unit norm $A-$tight frame satisfies [6]:

$$M = \sum_{i=1}^{M} \|\boldsymbol{\psi}_i\|_2^2 = \mathrm{tr}(\Psi^H \Psi) = \mathrm{tr}(\Psi \Psi^H) = am \tag{3}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. From the above, one can conclude that, for a UNTF, the tightness parameter $a = \frac{M}{m}$, which is nothing but the redundancy of the frame [3]. Unit norm tight frames have been used in the construction of signature sequences in code division multiple access systems [7, 8]. Moreover, they facilitate robust signal recovery in the presence of additive noise and erasures, and allow for stable reconstruction in communications related applications [9, 10, 11].

An equal norm tight frame is said to be an equiangular tight frame (ETF), if there exists a constant d such that $|\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle| = d$, for $1 \leq i < j \leq M$. Equiangular tight frames have been popular due to their use in sparse approximation [12], robust transmission [9, 13] and quantum computing [14].

A tight frame is said to be a unimodular tight frame (UMTF) if $|\psi_{ij}| = 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, M$, where ψ_{ij} denotes the (i, j)th element of $\Psi \in \mathbb{C}^{m \times M}$. If Ψ is a unimodular tight frame, then $\frac{1}{\sqrt{m}}\Psi$ is a unit norm tight frame, that is, a normalized UMTF is a UNTF. UMTFs have a wide range of applications in computer vision and image processing. For example, they have been applied in spectral analysis, audio and image processing [15, 16, 17]. In code-division multiple-access (CDMA) applications, tight frames whose individual vectors have low peak-to-average power

[1] In this paper, we use boldface lowercase letters to represent vectors. Normal font capital letters represent scalars or matrices, depending on the context.

ratio (PAPR) play an important role. Since every entry of a UMTF has unit modulus, the PAPR of each column equals one. Therefore, UMTFs are perhaps the most interesting example of frames with low PAPR [18, 19]. Also, applications of UMTFs in signal compression and compressed sensing are discussed in [20, 21]. UMTFs also offer the advantage of ease of implementation. The multiplication of a UMTF with a vector, when done in polar coordinates, involves about five floating point operations (flops) per entry. In contrast, an arbitrary matrix-vector multiplication involves six flops (in polar form) and eight flops (in cartesian form) per entry, i.e., the UMTFs offer about 20% to 60% reduction in the computational cost.

The mutual-coherence $\mu(\Psi)$ [22] of a given frame Ψ is the largest absolute inner product between different normalized columns of Ψ, that is,

$$\mu(\Psi) = \max_{1\leq\, i,j \leq\, M,\, i\neq j} \frac{|\ \langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle\ |}{\|\boldsymbol{\psi}_i\| \|\boldsymbol{\psi}_j\|}. \tag{4}$$

Here, $\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle \triangleq \boldsymbol{\psi}_i^H \boldsymbol{\psi}_j$ denotes the inner product between $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_j$, and $\boldsymbol{\psi}_i$ is the i-th column of Ψ. The mutual coherence characterizes the degree of correlation between the columns of a frame. An incoherent frame is one that has low correlation between any pair of columns. A lower bound on the minimal achievable correlation for any arbitrary frame, known as the Welch bound, is given by [23]

$$\mu(\Psi) \geq \sqrt{\frac{M-m}{m(M-1)}}. \tag{5}$$

Equiangular tight frames are the unit norm ensembles that achieve equality in the above Welch bound. However, the construction of unit norm tight frames and equiangular tight frames has proved to be notoriously difficult, and the few known design techniques are for a restricted set of frame dimensions.

The frame potential (total squared correlation) is defined as

$$\mathrm{FP}(\Psi) \triangleq \sum_{i=1}^{M} \sum_{j=1}^{M} |\ \langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle\ |^2. \tag{6}$$

For any $\Psi \in \mathbb{C}^{m\times M}$, the frame potential (FP) is a useful metric; it measures how close a frame is to being orthogonal. If $m \leq M$, in [24] the authors prove the following inequality:

$$\frac{M^2}{m} \leq \mathrm{FP}(\Psi) \leq M^2. \tag{7}$$

Further, they showed that:

Theorem 1. *Let M be fixed and consider the minimization of the frame potential among all collections of M points on the complex m dimensional unit sphere. Then,*

1. *Every local minimizer of the frame potential is also a global minimizer.*
2. *If $m \leq M$, then the minimum of the frame potential is $\frac{M^2}{m}$. The minimizers are UNTFs with M elements in $\mathbb{C}^m$.*

That is, by locally minimizing the frame potential, one obtains a UNTF with frame potential $\frac{M^2}{m}$.

Two techniques are known to provide general UNTF constructions. In [25], the authors start from a tight frame and by solving a differential equation, they approach a UNTF. In [26], the authors start from a unit norm frame and increase the degree of tightness using a gradient-descent-based algorithm. Relative primality of m and M is a condition assumed by both techniques, though in a weaker sense in [26]. In [27, 28] the authors present constructions for frames with the eignenvalues of the resulting Gram matrix $\Psi^H\Psi$ having a given spectrum and with a prescribed column norm. As for the construction of equiangular tight frames [29, 30, 31, 32], these frames exist only for certain frame dimensions such as $M \leq m^2$. In [33, 34], by using successive projections on the Gram matrix, incoherent UNTFs have been constructed. Most of these procedures do not yield UMTFs, or produce UMTFs only for a certain specific dimensions.

In [35], the authors constructed sets of sequences with good correlation properties. In particular, the design of sequence sets with good auto- and cross-correlation properties, via majorization-minimization (MM) methods was explored. Inspired by their work, in the present paper, we present an algorithm to construct unimodular tight frames, which are useful in compressed sensing applications. As mentioned earlier, the minimizers of the frame potential form unit norm tight frames, and a normalized UMTF is a UNTF. Further, recent results suggest that minimizing the FP is desirable in compressed sensing applications [36, 37, 38, 39, 40]. This motivates us to design UMTFs by minimizing the FP. The novel contributions of this work are summarized as follows:

- We propose an algorithm to find UMTFs by minimizing the frame potential using the MM method. The solution involves majorizing the objective function twice in a nested fashion, following which, we solve the innermost optimization problem in closed-form.

- We extend the algorithm to the case where the phase angles used in the sensing matrix are required to come from a finite set. Such a finite set constraint could arise due to hardware-specific implementation requirements, e.g., when only certain values of the phase angle are realizable. Our algorithm outputs a locally optimal solution to minimizing the frame potential subject to the phase angle constraints.

- We also present an extension of the algorithm to the case where sparsifying basis for the signal being compressively sensed is an arbitrary unitary matrix Γ. For example, in image processing applications, Γ can be taken to be the discrete cosine transform (DCT) matrix. In this case, our construction yields a matrix such that the effective dictionary used for sparse recovery is a UMTF, and therefore offers both the computational advantages as well as the superior recovery performance of UMTFs. We numerically show that the constructed UMTFs outperform their random and structured random counterparts such as the complex Gaussian, Bernoulli random and partial DFT matrices in terms of the sparse signal recovery performance, in a variety of sparsifying bases. We also show results quantifying the performance of the proposed sensing matrices for image reconstruction.

Thus, the construction presented in this paper provides a computationally simple and fast-converging method for finding frames that minimize the total coherence. These matrices are good sensing matrices for use in compressed sensing applications. Also, the algorithm can be used to generate frames of arbitrary dimensions, unlike the deterministic constructions proposed in past literature [41]. Finally, as mentioned earlier, UMTFs can be implemented using fewer computational resources compared to arbitrary sensing matrices, making them attractive for implementation.

The rest of this paper is organized as follows. In Section 2, we present the basic principle underlying MM methods. The key novelty involved in an MM approach is to find an upper bound on the cost function that is tight at the current iterate and easy to optimize. We present this in Section 3, which leads to our proposed algorithm for constructing UMTFs. In Section 4, we relate the problem of minimizing the frame potential to that of minimizing the total coherence of the matrix, which motivates their use in compressed sensing applications. In Section 5, we present simulation results illustrating the performance of our algorithm both for sparse signal recovery and image reconstruction. In the last section, Section 6, we present our concluding remarks.

2. The Majorization-Minimization (MM) method

In this section, we give a brief overview of the MM method. The MM method is a generalization of the well known expectation maximization (EM) algorithm [42], and has been successfully used in many applications in statistical computation, variable selection [43], and signal/image processing [44, 45]. For an excellent tutorial on MM algorithms, we refer the readers to [46] and the references therein.

The general idea of MM algorithms is as follows. Suppose we want to minimize a cost function $f(\mathbf{x})$ over $\mathcal{X} \subset \mathbb{C}^m$. Instead of minimizing $f(\mathbf{x})$ directly, the MM approach optimizes a sequence of approximate objective functions $g(\mathbf{x}, \mathbf{x}^{(k)})$ that majorize $f(\mathbf{x})$. The function $g(\mathbf{x}, \mathbf{x}^{(k)})$ is said to majorize $f(\mathbf{x})$ at the point $\mathbf{x}^{(k)}$ if

$$f(\mathbf{x}) \leq g(\mathbf{x}, \mathbf{x}^{(k)}), \forall \mathbf{x} \in \mathcal{X}, \text{ and } f(\mathbf{x}^{(k)}) = g(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}). \tag{8}$$

That is, $g(\mathbf{x}, \mathbf{x}^{(k)})$ is an upper bound of $f(\mathbf{x})$ over $\mathcal{X}$ and coincides with $f(\mathbf{x})$ at $\mathbf{x}^{(k)}$. The MM algorithm corresponding to this majorization function g starts with an arbitrary feasible point $\mathbf{x}^0$ and produces a sequence $\{\mathbf{x}^{(k)}\}$ according to the following update rule:

$$\mathbf{x}^{(k+1)} = \text{arg min}_{\mathbf{x} \in \mathcal{X}}\ g(\mathbf{x}, \mathbf{x}^{(k)}), \tag{9}$$

where the point $\mathbf{x}^{(k)}$ is generated by the algorithm at the k^{th} iteration.

It is easy to show that the above iterative scheme decreases the value of f monotonically in each iteration, that is,

$$f(\mathbf{x}^{(k+1)}) \leq g(\mathbf{x}^{(k+1)}, \mathbf{x}^{(k)}) \leq g(\mathbf{x}^{(k)}, \mathbf{x}^{(k)}) = f(\mathbf{x}^{(k)}), \tag{10}$$

where the first and last inequalities follow from (8) while the middle one follows from (9). Due to the monotonic decrease property, MM algorithms are guaranteed to converge to a stationary point from any initialization [47]. However, when the cost function has multiple local (or global) optima, the solution to which the MM algorithm converges depends on the initialization. As will be empirically shown in this paper, in certain applications, a random initialization often yields better solutions than a hand-picked, deterministic initialization. Therefore, the choice of the initialization affects the performance of MM based algorithms. The key creative step in the MM algorithms is to find a majorization function of the cost function such that the majorized problem is easy to solve. In the next section, we present our proposed the algorithm for constructing UMTFs.

3. Construction of Unimodular Tight Frames

In this section, we propose an MM-based algorithm for the construction of unimodular tight frames. As mentioned earlier, unimodular tight frames are minimizers of the frame potential. Therefore, one can generate them by solving the following optimization problem:

$$\underset{\{\boldsymbol{\psi}_l\}_{l=1}^M}{\arg\min} \sum_{i,j=1}^{M} |\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle|^2 \text{ s.t. } |\psi_{ij}| = 1, \forall i \text{ and } j. \tag{11}$$

Let us first stack the frame $\{\boldsymbol{\psi}_l \in \mathbb{C}^m\}_{l=1}^M$ together and denote it by ψ as $\psi = [\boldsymbol{\psi}_1^T, \ldots, \boldsymbol{\psi}_M^T]^T$, where $(\cdot)^T$ is the transpose of the vector $(\cdot)$.[2] Then, we have

$$\boldsymbol{\psi}_l = S_l \psi, \tag{12}$$

where S_l is an $m \times mM$ block selection matrix defined as

$$S_l = [0_{m\times(l-1)m}, I_m, 0_{m\times(M-l)m}],$$

and I_m is the identity matrix of size m. From (12), we have,

$$\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle = \psi^H S_j^H S_i \psi, \tag{13}$$

and then

$$\begin{aligned} |\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle|^2 &= |\psi^H S_j^H S_i \psi|^2 \\ &= |\mathrm{tr}(\psi^H S_j^H S_i \psi)|^2 \\ &= |\mathrm{vec}(\psi\psi^H)^H \mathrm{vec}(S_j^H S_i)|^2, \end{aligned} \tag{14}$$

where $\mathrm{vec}(\cdot)$ is a column vector consisting of all the columns of the matrix $(\cdot)$, stacked vertically. By using (14), the minimization problem in (11) can be written as

$$\begin{aligned} &\underset{\psi \in \mathbb{C}^{mM}}{\arg\min} \ \mathrm{vec}(\psi\psi^H)^H L \mathrm{vec}(\psi\psi^H) \\ &\text{subject to } |\psi_i| = 1, i = 1, \ldots, mM, \end{aligned} \tag{15}$$

where ψ_i denotes the ith entry of ψ, and

$$L = \sum_{i,j=1}^{M} \mathrm{vec}(S_j^H S_i) \mathrm{vec}(S_j^H S_i)^H. \tag{16}$$

By using the following lemmas [48, 49], we can majorize the objective function in (15).

Lemma 2. *Let $P_{m\times m}$ be a real symmetric non-negative matrix. Then the problem*

$$\min_{\mathbf{b}} \mathbf{b}^T \mathbf{1}_m \ \textit{subject to diag}(\mathbf{b}) \succeq P \tag{17}$$

admits the optimal solution $\mathbf{b}^H = P\mathbf{1}_m$, where diag$(\cdot)$ is a diagonal matrix formed with the vector $(\cdot)$ as its principal diagonal, and $\mathbf{1}_m$ is an all-ones vector of dimension m. In (17), the notation $A \succeq B$ denotes that the matrix $A - B$ is positive semi-definite.

[2] Note that, with a slight abuse of notation, we use ψ to denote the stacked version of $\{\boldsymbol{\psi}_l\}_{l=1}^M$.

Lemma 3. *Let P be an $m \times m$ matrix and ψ be a vector in $\mathbb{C}^m$ with $|\psi_i| = 1, \forall i$. Then $P \odot (\psi\psi^H)$ and P share the same set of eigenvalues. Here, $\odot$ is the Hadamard product of two matrices.*

Lemma 4. *Let P be an $m \times m$ Hermitian matrix and Q be another $m \times m$ Hermitian matrix such that $Q \succeq P$. Then for any point $\mathbf{x}_0 \in \mathbb{C}^m$, the quadratic function $\mathbf{x}^H P \mathbf{x}$ is majorized by $\mathbf{x}^H Q \mathbf{x} + 2Re(\mathbf{x}^H (P - Q)\mathbf{x}_0) + \mathbf{x}_0^H (Q - P)\mathbf{x}_0$ at $\mathbf{x}_0$. Here, $Re(\cdot)$ denotes the real part of $(\cdot)$.*

It is easy to see that L in (16) is a nonnegative real symmetric matrix. By using lemma 2, it can be shown that $\text{diag}(\mathbf{b}) \succeq L$, where $\mathbf{b} = L\mathbf{1}_{m^2M^2}$. Therefore, for a given $\Psi^{(l)}$ at iteration l, by using lemma 4, the objective function in (15) can be majorized by the following function at $\psi^{(l)}$:

$$\begin{aligned} g_1(\psi, \psi^{(l)}) &= \text{vec}(\psi\psi^H)^H \text{diag}(\mathbf{b}) \text{vec}(\psi\psi^H) + 2\text{Re}(\text{vec}(\psi\psi^H)^H (L - \text{diag}(\mathbf{b})) \text{vec}(\psi^{(l)}{\psi^{(l)}}^H)) \\ &+ \text{vec}(\psi^{(l)}{\psi^{(l)}}^H)^H (\text{diag}(\mathbf{b}) - L) \text{vec}(\psi^{(l)}{\psi^{(l)}}^H). \end{aligned} \tag{18}$$

Since the elements of ψ are of unit modulus, it is easy to see that the first term in (18) is a constant. Further, since $\psi^{(l)}$ is given by the previous iteration, the third also becomes a constant. After ignoring the constant terms, the majorized problem of (15) is given by

$$\begin{aligned} \min_{\psi} \quad & \text{Re}(\text{vec}(\psi\psi^H)^H (L - \text{diag}(\mathbf{b})) \text{vec}(\psi^{(l)}{\psi^{(l)}}^H)) \\ & \text{subject to } |\psi_i| = 1, i = 1, \ldots, mM. \end{aligned} \tag{19}$$

By substituting L from (16) in (19), we can write the first term as

$$\begin{aligned} &\text{Re}(\text{vec}(\psi\psi^H)^H L \text{vec}(\psi^{(l)}{\psi^{(l)}}^H)) = \\ &\sum_{i,j=1}^{M} \text{Re}(\text{vec}(\psi\psi^H)^H \text{vec}(S_j^H S_i) \text{vec}(S_j^H S_i)^H \text{vec}(\psi^{(l)}{\psi^{(l)}}^H)) \\ &= \sum_{i,j=1}^{M} \text{Re}(\text{tr}(\psi\psi^H S_j^H S_i)(\text{vec}(\psi^{(l)}{\psi^{(l)}}^H)^H \text{vec}(S_j^H S_i))^H) \\ &= \sum_{i,j=1}^{M} \text{Re}\left(\text{tr}(\psi^H S_j^H S_i \psi)\left(\text{tr}({\psi^{(l)}}^H S_j^H S_i \psi^{(l)})\right)^H\right) \\ &= \sum_{i,j=1}^{M} \text{Re}\left((\psi^H S_j^H S_i \psi)({\psi^{(l)}}^H S_i^H S_j \psi^{(l)})\right) \\ &= \sum_{i,j=1}^{M} \text{Re}\left(\left\langle \boldsymbol{\psi}_j^{(l)}, \boldsymbol{\psi}_i^{(l)} \right\rangle (\psi^H S_j^H S_i \psi)\right) \end{aligned} \tag{20}$$

and the second term in (19) can also be written as

$$
\begin{aligned}
&\mathrm{Re}(\mathrm{vec}(\psi\psi^H)^H \mathrm{diag}(\mathbf{b})\mathrm{vec}(\psi^{(l)}\psi^{(l)^H}))\\
&\quad= \mathrm{Re}\left(\mathrm{vec}(\psi\psi^H)^H \left(\mathbf{b} \odot \mathrm{vec}(\psi^{(l)}\psi^{(l)^H})\right)\right)\\
&\quad= \mathrm{Re}\left(\mathrm{vec}(\psi\psi^H)^H \mathrm{vec}\left(\mathrm{mat}\left((L\,\mathbf{1}_{m^2M^2}) \odot \mathrm{vec}(\psi^{(l)}\psi^{(l)^H})\right)\right)\right)\\
&\quad= \mathrm{Re}\left(\mathrm{tr}\left((\psi\psi^H)\mathrm{mat}\left((L\,\mathbf{1}_{m^2M^2}) \odot \mathrm{vec}(\psi^{(l)}\psi^{(l)^H}\right)\right)\right)\\
&\quad= \mathrm{Re}\left(\psi^H \mathrm{mat}\left((L\,\mathbf{1}_{m^2M^2}) \odot \mathrm{vec}(\psi^{(l)}\psi^{(l)^H})\right)\psi\right)\\
&\quad= \mathrm{Re}\left(\psi^H \left(\mathrm{mat}(L\,\mathbf{1}_{m^2M^2}) \odot (\psi^{(l)}\psi^{(l)^H})\right)\psi\right)\\
&\quad= \mathrm{Re}\left(\psi^H \left(\mathrm{mat}\left(\sum_{i,j=1}^{M} m\ \mathrm{vec}(S_j^H S_i)\right)\odot(\psi^{(l)}\psi^{(l)^H})\right)\psi\right)\\
&\quad= \mathrm{Re}\left(\psi^H \left(\left(\sum_{i,j=1}^{M} m\ S_j^H S_i\right)\odot(\psi^{(l)}\psi^{(l)^H})\right)\psi\right)\\
&\quad= \mathrm{Re}\left(\psi^H \left((\mathbf{1}_{M\times M} \otimes mI_m) \odot (\psi^{(l)}\psi^{(l)^H})\right)\psi\right),
\end{aligned} \tag{21}
$$

where $\mathrm{mat}(\cdot)$ is the inverse operation of $\mathrm{vec}(\cdot)$, $\otimes$ denotes the Kronecker product, and $\mathbf{1}_{M\times M}$ is the constant matrix of size $M \times M$ with all entries being equal to 1.

Using (20) and (21), the minimization problem in (19) can be written as follows:

$$
\begin{aligned}
&\min_{\psi}\ \ \psi^H \left(S - (R \odot (\psi^{(l)}\psi^{(l)^H}))\right)\psi \\
&\text{subject to } |\psi_i| = 1;\ i = 1, \ldots, mM,
\end{aligned} \tag{22}
$$

where

$$
R \triangleq \mathbf{1}_{M\times M} \otimes mI_m \text{ and } S = \sum_{i,j=1}^{M} \left\langle \boldsymbol{\psi}_j^{(l)}, \boldsymbol{\psi}_i^{(l)} \right\rangle (S_j^H S_i) \tag{23}
$$

Since R and S are Hermitian matrices, we have removed the $\mathrm{Re}(\cdot)$ operator from (22). The minimization problem in (22) is still difficult to directly solve. Hence, we majorize the objective function at $\psi^{(l)}$ to further simplify the problem at every iteration. To construct a majorization function for the objective function in (22), we need to find a matrix Q such that $(S - (R \odot (\psi^{(l)}\psi^{(l)^H})) \preceq Q$ and the obvious choice may be $Q = \lambda_{\max}(S - (R \odot (\psi^{(l)}\psi^{(l)^H}))I$. From Lemma 3, we have the following:

$$
\begin{aligned}
\lambda_{\max}(S - (R \odot (\psi^{(l)}\psi^{(l)^H}))) &\leq \lambda_{\max}(S) - \lambda_{\min}(R),\\
&\leq \|S\|_\infty - \lambda_{\min}(R),
\end{aligned} \tag{24}
$$

where $\lambda_{\max}(\cdot), \lambda_{\min}(\cdot)$ are the maximum and minimum eigenvalues of $(\cdot)$, respectively, and $\|\cdot\|_\infty$ is the infinity norm of a matrix, i.e., its maximum absolute row sum.

Since $M \geq m$ and the eigenvalues of $A \otimes B$ are the product of eigenvalues of A and B, we have $\lambda_{\min}(R) = 0$. Therefore,

$$
\lambda_{\max}\left(S - \left(R \odot \left(\psi^{(l)}\psi^{(l)^H}\right)\right)\right) \leq \|S\|_\infty. \tag{25}
$$

Algorithm 1 MM algorithm for generating UMTFs

Input: Frame dimensions m and M
Output: UMTF $\{\boldsymbol{\psi}_i\}_{i=1}^M$ in $\mathbb{C}^m$ of size M

Initialize $\Psi^0 = [\boldsymbol{\psi}_1^0, \ldots, \boldsymbol{\psi}_M^0]_{m\times M}$ randomly such that $|\psi_{iq}^{(0)}| = 1, i = 1, \ldots, M$ and $q = 1, \ldots m$.
Require $S_l = [0_{m\times(l-1)m}, I_m, 0_{m\times(M-l)m}]_{m\times mM}$, and $R = \mathbf{1}_{M\times M} \otimes mI_m$.
repeat
    Compute frame potential $\mathrm{FP}(\Psi^{(r)})$ using (6).
    Set $\psi^{(r)} = [\boldsymbol{\psi}_1^{(r)^T}, \ldots, \boldsymbol{\psi}_M^{(r)^T}]^T$
    Compute $S^{(r)} = \sum_{i,j=1}^M \left\langle \boldsymbol{\psi}_j^{(r)}, \boldsymbol{\psi}_i^{(r)} \right\rangle (S_j^H S_i)$.
    Compute $\phi^{(r)} = (S^{(r)} - (R \odot (\psi^{(r)}\psi^{(r)^H}))\psi^{(r)} - \|S^{(r)}\|_\infty \psi^{(r)}$.
    Calculate $\psi_i^{(r+1)} = e^{j\arg(-\phi_i^{(r)})}; i = 1, \ldots, mM$.
    $\Psi^{(r+1)} =$ reshape$(\psi^{(r+1)}, m, M)$
    $r \longleftarrow r + 1$
until convergence

Now by choosing $Q = \|S\|_\infty I$ in Lemma 4, the objective function in (22) is majorized by

$$g_2(\psi, \psi^{(l)}) = \|S\|_\infty \psi^H \psi + 2\mathrm{Re}\left(\psi^H \left(S - (R \odot (\psi^{(l)}\psi^{(l)^H})) - \|S\|_\infty I\right) \psi^{(l)}\right) + \psi^{(l)^H} \left(\|S\|_\infty I - S + (R \odot (\psi^{(l)}\psi^{(l)^H}))\right) \psi^{(l)}. \tag{26}$$

After ignoring the constant terms, the majorized problem of (22) is given by

$$\min_{\psi} \ \mathrm{Re}(\psi^H \phi) \tag{27}$$
$$\text{subject to } |\psi_i| = 1; i = 1, \ldots, mM,$$

where $\phi = (S - (R \odot (\psi^{(l)}\psi^{(l)^H})))\psi^{(l)} - \|S\|_\infty \psi^{(l)}$. The minimization problem in (27) is separable in the elements of ψ. The closed-form solution is as follows:

$$\psi_i = e^{j\arg(-\phi_i)}; i = 1, \ldots, mM, \tag{28}$$

where $\arg(\cdot)$ represents the argument (phase) of $(\cdot)$. This concludes the development of the algorithm for constructing unimodular tight frames. The algorithm minimizes the frame potential using a majorization-minimization approach. The pseudo-code for the proposed approach is presented in Algorithm 1. In the next subsection, we extend the algorithm to the case where the phase angles are restricted to a finite set of feasible phases. In the Sec. 3.3, we discuss the convergence of proposed algorithm.

3.1. Extension to the case of Finite Set of Feasible Phase Angles

In many practical applications, not all phase angles are practically realizable, and a designer may be required to design the sensing matrix such that the phase angles belong to a finite set denoted by $\mathcal{A}$.[3] Then, the optimization problem (11) can be modified as

$$\arg\min_{\{\boldsymbol{\psi}_l\}_{l=1}^M} \sum_{i,j=1}^M |\langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle|^2 \text{ s.t. } |\psi_{ij}| = 1, \arg(\psi_{ij}) \in \mathcal{A}, \forall i \text{ and } j. \tag{29}$$

[3] For example, $\mathcal{A} = \{0, \pi/2, \pi, 3\pi/2\}$.

The solution to the above problem can be obtained using exactly the same steps as described above, except that the final-stage optimization problem becomes

$$\min_{\psi} \ \mathrm{Re}(\psi^H \phi) \tag{30}$$
$$\text{subject to } |\psi_i| = 1, \arg(\psi_i) \in \mathcal{A}; i = 1, \ldots, mM.$$

The above objective function is clearly separable in each of the variables ψ_i, and therefore, following (28), its solution can be obtained by simply computing $\mathrm{Re}(\psi_i^H \phi_i)$ with the phase in ψ_i set equal to the two nearest angles in the set $\mathcal{A}$ to $\arg(-\phi_i)$, and picking the solution with the lower value. Thus, it is straightforward to extend our solution to the case where the phase angles are restricted to a finite set of feasible angles.

3.2. Per-Iteration Complexity

In this sub-section, we discuss the per-iteration complexity of Algorithm 1. In Step 4 of the algorithm, the computation of the frame potential is of complexity $M^2 m$ floating point operations (flops). In Step 6, computing $S^{(r)}$ is of complexity M^2 flops. Step 7 is of complexity $(Mm)^2$ as it involves computing $\psi^{(r)}\psi^{(r)^H}$ which is of size $mM \times mM$. The other steps in the algorithm are of negligible complexity. Thus, the overall per-iteration complexity of the algorithm is $\mathcal{O}((Mm)^2)$.

3.3. Convergence Properties

Since the frame potential is a bounded, continuous function on a compact set, our algorithm is guaranteed to converge to a stationary point of the frame potential from any initialization by virtue of the fact that it is an MM procedure (see (10)) [47]. From Theorem 1, when the algorithm converges to a frame potential value of M^2/m, it is guaranteed to be a UNTF.[4] Finally, as stated earlier, a normalized UMTF is a UNTF, and hence, the minimizers are precisely normalized UMTFs.

We numerically illustrate the convergence of our algorithm in Figure 1, which shows the convergence of the frame potential to its lower bound, as a function of the number of iterations, for UMTFs of size $8 \times 64, 18 \times 96$ and 32×128. We choose these dimensions because $M^2/m = 512$, and hence the normalized UMTF achieves an FP of 512, in all three cases. We observe that our algorithm converges exponentially fast to the lower bound on frame potential in (7), for all dimensions of the normalized UMTFs.

So far, we discussed how one can construct UMTFs using majorization-minimization methods and the convergence properties of our algorithm. The construction of frames that allow for sparse coding and representation is of recent research interest [50]. In the next section, we discuss the applicability of our algorithm to construct sensing matrices for sparse coding.

4. Application to Compressed Sensing

In recent years, Compressed Sensing (CS) and sparse representation have become a powerful tool for efficiently compressing and processing data. One of the central problems in CS is the

[4] Empirically, we find that, upon convergence, the FP of the solution always equals M^2/m, for all matrix dimensions tested.

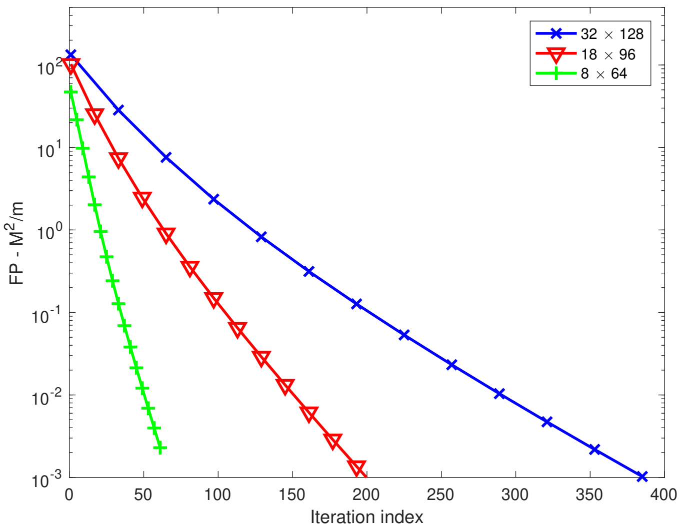


Figure 1: Difference between the FP of the algorithm and the optimal FP, as a function of the iteration index, for normalized UMTFs of size $8 \times 64, 18 \times 96$ and 32×128.

construction of *sensing matrices* $\Psi \in \mathbb{C}^{m\times M}$ such that an arbitrary s-sparse vector $\mathbf{u} \in \mathbb{C}^{M}$ can be efficiently reconstructed using the underdetermined linear projection $\mathbf{y} = \Psi\mathbf{u}$. The vector $\mathbf{u}$ is said to be s-sparse when it can be represented using at most s nonzero coefficients in an orthogonal vector representation. That is, $\mathbf{u}$ can be decomposed as $\mathbf{u} = \Gamma\boldsymbol{\alpha}$, where the unitary matrix $\Gamma \in \mathbb{C}^{M\times M}$ is the sparsifying basis and $\boldsymbol{\alpha} \in \mathbb{C}^{M}$ has $s-$non zero entries, i.e., $\|\boldsymbol{\alpha}\|_0 \triangleq |\{i : \alpha_i \neq 0\}| \leq s$.

One can find $\mathbf{u}$ from its underdetermined linear projection $y = \Psi\mathbf{u} = \Psi\Gamma\boldsymbol{\alpha}$ by solving the following l_0-minimization problem:

$$\hat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{\alpha}} \|\boldsymbol{\alpha}\|_0 \text{ subject to } \mathbf{y} = \Phi\boldsymbol{\alpha}, \tag{31}$$

where $\Phi \triangleq \Psi\Gamma$, and then computing $\hat{\mathbf{u}} = \Gamma\hat{\boldsymbol{\alpha}}$ as the estimate of $\mathbf{u}$. The l_0-minimization problem (31) is an NP hard problem [51]. Candes et al. [51] have proposed the following l_1-minimization problem in place of (31), making it a computationally tractable linear program:

$$\hat{\boldsymbol{\alpha}} =: \arg\min_{\boldsymbol{\alpha}} \|\boldsymbol{\alpha}\|_1 \text{ subject to } \mathbf{y} = \Phi\boldsymbol{\alpha}. \tag{32}$$

Here, $\|\boldsymbol{\alpha}\|_1$ denotes the l_1-norm of the vector $\boldsymbol{\alpha} \in \mathbb{C}^{M}$.

One of the important breakthroughs in the CS literature is the characterization of the conditions under which problems (31) and (32) admit the same solution. The concept of coherence provides a condition for equivalence between them, which is stated as follows [12]:

Theorem 5. *An arbitrary $s-$sparse signal $\boldsymbol{\alpha}$ can be uniquely recovered from $\mathbf{y} = \Phi\boldsymbol{\alpha}$ as a solution to (32) provided*

$$s < \frac{1}{2}\left(1 + \frac{1}{\mu(\Phi)}\right). \tag{33}$$

The mutual coherence plays an important role in the performance of the algorithms for finding sparse solutions in (31) [52]. The smaller $\mu(\Phi)$, the higher the bound on the sparsity of $\boldsymbol{\alpha}$. Hence, the construction of incoherent frames is an active area of research in the field of frame theory and CS. In the case of incoherent frame design, perhaps the most influential work is [53], which proposes a design procedure based on the alternating projection method. The authors in [34] improved the shrinkage operation proposed in [53] to produce frames with better incoherence, based on averaged projections. In [54], the authors constructed incoherent frames using convex optimization techniques. An efficient refinement to solve the optimization problem in [54], using numerical weights, was proposed in [55]. In [56], the authors proposed a new algorithm to construct incoherent frames by minimizing the maximum absolute pairwise correlations (mutual coherence) of the frame vectors based on an alternating minimization penalty method. These algorithms involve matrix inversions and/or the computation of the singular value decomposition (SVD) in the iterations, which can be computationally expensive for constructing large dimensional matrices. Also, alternating minimization often exhibits very slow convergence. Our proposed algorithm, on the other hand, does not involve matrix inversions or SVD, and converges fast as it is based on the MM principle.

Note that (33) is a sufficient condition for the successful recovery of sparse vectors via ℓ_1 minimization, it does not reflect the *average* recovery ability of sparse approximation methods. Nonetheless, it does suggest that recovery may be improved when Φ is as orthogonal as possible [38]. Motivated by these observations, the authors in [36, 37, 38] proposed designing the sensing matrix Ψ by minimizing $\|\Phi'\Phi - I\|_F^2$, where $\|.\|_F$ is the Frobenius norm of a matrix and I is the $M \times M$ identity matrix. That is, they constructed Φ by solving the following optimization problem:

$$\arg\min_{\Psi} \|\Phi'\Phi - I\|_F^2 = \arg\min_{\Psi} \|\Gamma'\Psi'\Psi\Gamma - I\|_F^2. \tag{34}$$

That is, rather than minimizing coherence $\mu(\Phi)$, they minimize the sum of the squared inner products of all distinct pairs of columns in Φ, which is the same as the total coherence $\mu^t(\Phi) \triangleq \sum_{i \neq j} |\phi_i'\phi_j|^2$ [38]. The following proposition states the equivalence between the problems (11) and (34), when the sensing matrix is unimodular and the sparsifying basis is canonical (that is, $\Gamma = I$). We omit its proof as it is straightforward from elementary algebraic manipulation.

Proposition 6. *If the sensing matrix Ψ is unimodular and sparsifying basis is canonical then solving (34) is equivalent to solving (11).*

We have thus established the equivalence between the problems of constructing unimodular tight frames and compressed sensing matrices, when sensing matrix Ψ is unimodular and the sparsifying basis is the canonical one. We now discuss the case where the sparsifying basis is not the canonical basis.

4.1. Extension to Sparsifying Basis Other than Canonical Basis

As mentioned above, in many applications, the signal that is being compressively sampled is not sparse in the canonical basis, but is sparse in the basis Γ. For example, images are known to be sparse in the DCT domain, and hence the signal $\mathbf{u}$ that is compressively sampled can be expressed as $\mathbf{u} = \Gamma\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a sparse vector. In this case, we can reuse the construction derived above. Following [33], we solve a least squares problem to find a matrix Ψ' such that $\Psi = \Psi'\Gamma$ in order to compute the sensing matrix Ψ'. Then, the effective matrix used in the recovery algorithm becomes

$\Psi'\Gamma$ which is a UMTF. Hence, it enjoys the benefits of low total coherence and low computational complexity for the recovery algorithms.

In the next section, we illustrate the performance of the proposed normalized unimodular tight frames via simulations.

5. Simulation Results

In this section, we compare the sparse vector recovery performance of matrices constructed using our method, the standard complex Gaussian (with entries drawn from $\mathcal{N}(0, \frac{1}{m})$), Bernoulli (with entries $\phi_{ij} = \pm\frac{1}{\sqrt{m}}$, each with probability $\frac{1}{2}$) random matrices and partial DFT matrices (randomly selected rows from the full DFT matrix), via numerical simulations. We consider these alternatives as the other constructions in the literature do not yield UMTFs for the dimensions we consider in this paper. We also consider the canonical, Fourier, Haar-wavelet and DCT as the sparsifying basis for the comparison. For each sparsity level s, noiseless compressive measurements from 1000 $s-$sparse signals $\mathbf{x}$ with nonzero indices chosen uniformly randomly and the entries drawn from $\mathcal{N}(0, 1)$ are considered for obtaining the average recovery performance.

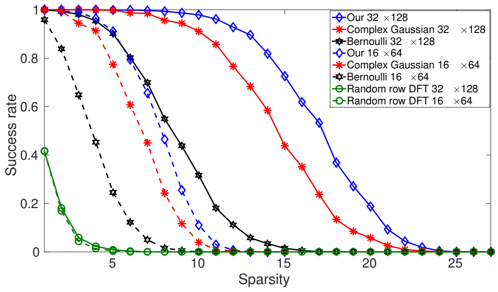
We consider the orthogonal matching pursuit (OMP) [57] algorithm for sparse vector recovery. Let $\tilde{\mathbf{x}}$ denote the recovered solution. For purposes of comparing the solutions, the signal-to-noise ratio (SNR) of $\mathbf{x}$ is defined as

$$\text{SNR}(\mathbf{x}) = 10 \log_{10} \left(\frac{\|\mathbf{x}\|_2}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2} \right) \text{ dB}.$$

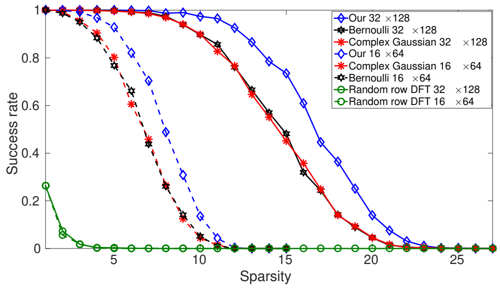
The recovery is considered successful if $\text{SNR}(\mathbf{x}) \geq 100$ dB. The success probability of recovering 64 and 128 length sparse vectors using 16 and 32 measurements, respectively, is plotted against the sparsity level in Figure 2 for the OMP algorithm for sparse recovery. We observe that the UMTFs constructed from our method outperform complex Gaussian and Bernoulli random matrices in all sparsifying bases. When compared with the partial DFT matrix, the performance is similar in the canonical and Haar-wavelet domains, while the proposed construction offers significantly better performance in the DCT and DFT domains.

5.1. Phase Transition

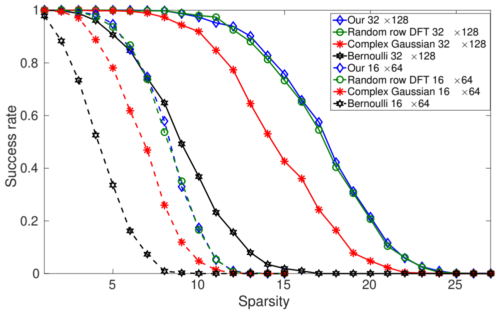
The phase transition diagrams depict the largest k (with fixed m and M) for faithfully recovering $k-$sparse vectors. Figure 3 compares the phase transition curves for matrices formed by our method with the standard complex Gaussian, Bernoulli random and partial DFT matrices, for different values of $\delta = \frac{m}{M}$, with $m = 8, 12, 16, 20, 24, 28, 32, 36, 40$ and $M = 64$ in DCT, Fourier, Haar-wavelets and Canonical domains. In Fig. 3, successful reconstruction is not possible in the region above the curve, while the recovery is successful in the region below the curve. We generate the phase transition curve by finding the largest sparsity k such that the probability of successful recovery is at least 90 percent, for each value of $\delta = \frac{m}{M}$. From the plot, we observe that the matrices constructed using our method outperform the complex Gaussian, Bernoulli random and partial DFT matrices in the DCT and DFT domains, whereas, they give similar performance as the partial DFT and outperform the complex Gaussian and Bernoulli random matrices in the Haar wavelet and canonical bases.



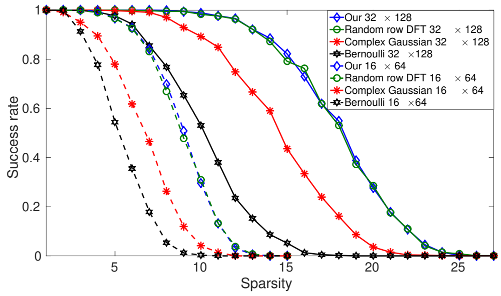
(a) DCT basis



(b) DFT basis

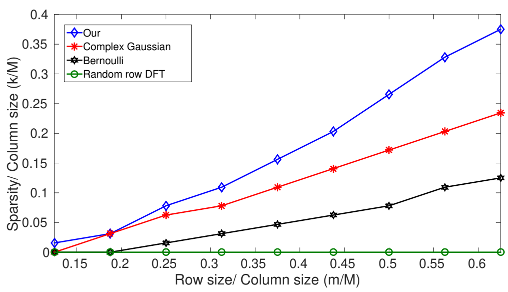


(c) Haar wavelet basis

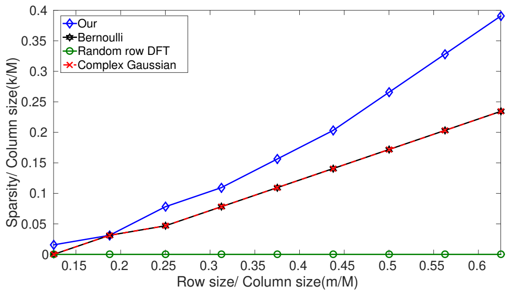


(d) Canonical basis

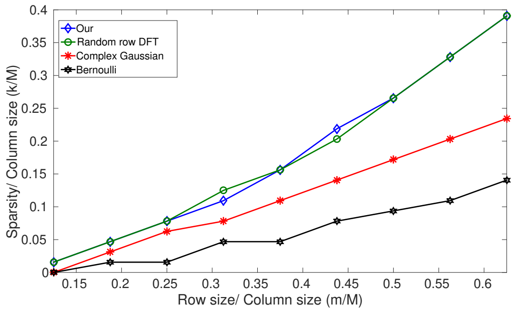
Figure 2: Comparison of the reconstruction performances of the matrices constructed from our method, complex Gaussian random, Bernoulli random and partial DFT matrices of size 16×64 and 32×128 by the OMP algorithm in different sparsifying bases.



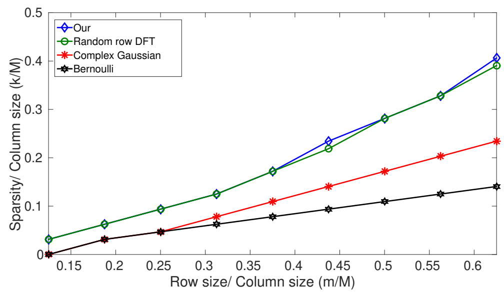
(a) DCT basis



(b) DFT basis



(c) Haar wavelet basis



(d) Canonical basis

Figure 3: Comparison of the matrices constructed using our method, complex Gaussian random, Bernoulli random and partial DFT matrices through the phase transition characteristics.

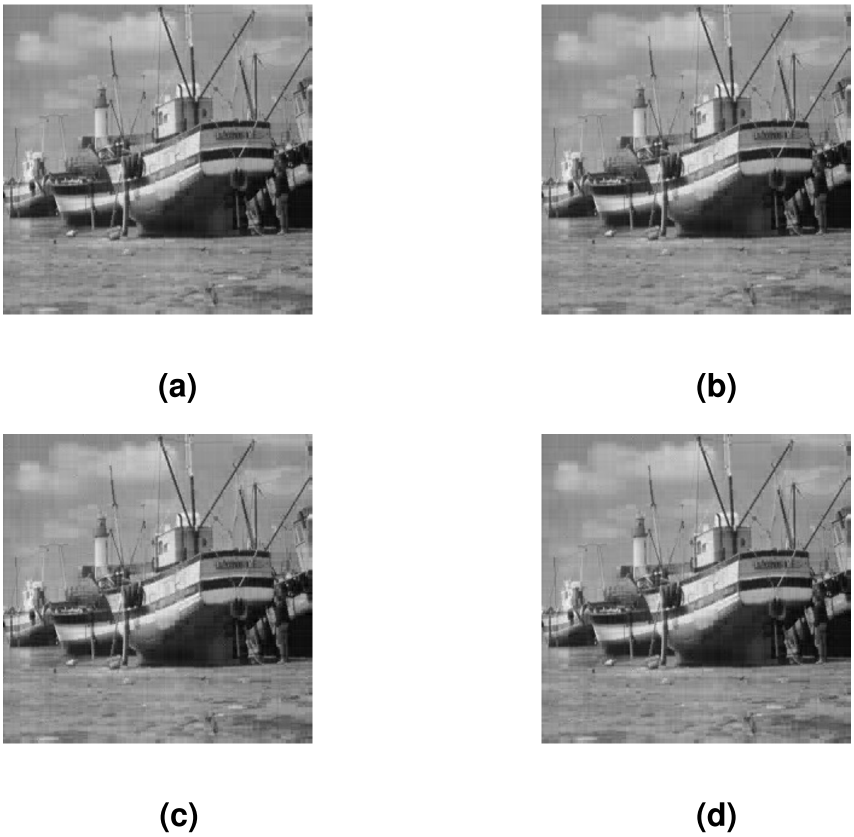


Figure 4: For the original image (a) of size 256×256, the image in (b) is reconstructed via the matrix constructed from our method; (c) and (d) were obtained via the corresponding complex Gaussian and partial DFT matrices, respectively.

5.2. Reconstruction of Images

In this subsection, we demonstrate the relative performances of the complex Gaussian, partial DFT, and the matrix obtained using our method in the context of image reconstruction from lower dimensional patches. The image is divided into smaller patches of equal size and the vectorized versions of each patch is sparsified by computing its Haar wavelet transform and retaining a pre-determined fraction of its wavelet coefficients, keeping the largest and setting the rest to zero. For each patch, 37.5% compressed measurements are taken using complex Gaussian, partial DFT and the matrix obtained by our method as measurement matrices. The images are reconstructed using the OMP algorithm. Figure 4 shows the reconstructed images. The associated reconstruction accuracy in terms of SNR are 14.77, 19.34 and 19.55 dB for the complex Gaussian, partial DFT and our construction, respectively. Thus, the matrices constructed from our method outperform the complex Gaussian matrix by about 4.8 dB and marginally outperform the partial DFT matrix.

6. Concluding Remarks

In this work, we presented the construction of uni-modular tight frames (UMTFs) by solving the optimization problem of minimizing the frame potential via a Majorization-Minimization (MM) approach. This approach allows one to construct UMTFs of arbitrary dimensions. We showed that minimizing the frame potential is equivalent to minimizing the total coherence when the frame is unimodular, which provided justification for using the proposed method to construct good measurement matrices for compressed sensing applications. Further, UMTFs are particularly attractive from an implementation perspective due to the low complexity of the multiplications involved. The resulting UMTF was shown to outperform complex Gaussian and partial DFT sensing matrices of the same size via numerical experiments in a variety of sparsifying bases. Future work could consider developing a theoretical understanding of the superior performance offered by the MM-based approach presented in this work, particularly in non-canonical bases. It would also be interesting to incorporate the MM-based procedure into dictionary learning algorithms to produce low-coherence learned dictionaries for specific applications.

Acknowledgments

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