Analysis of Error Probability with Maximum Likelihood Detection over Discrete-Time Memoryless Noncoherent Rayleigh Fading Channels

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Outline

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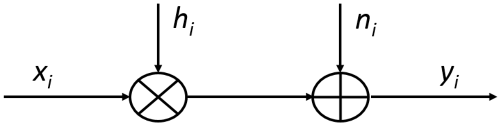
1. Motivation
2. System Model
3. Error Probability Analysis
4. Validation
5. Optimal Signal Constellation
6. Concluding Remarks

A Fast Fading Channel Model

- The discrete-time memoryless noncoherent Rayleigh fading channel (DTM-NRFC) well represents the fast fading mobile wireless channel
  - Coherence time ≈ symbol duration
  - Coherence bandwidth ≈ signal bandwidth
- Neither the transmitter nor the receiver has the channel state information
- Fading process: i.i.d. zero-mean complex-Gaussian
- Very pessimistic model indeed!
- Can lead to robust receivers



DTM-NRFC: Known Results

- [Taricco, Elia; 1997]:
  - Capacity bounds for very low and very high SNRs
  - Capacity $\propto \log \log \text{SNR}$
- [Abou-Faycal, Trott, Shamai; 2001]:
  - Capacity is achieved by a discrete constellation
  - One of the mass points is at the origin
- [Rezki, Haccoun, Gagnon; 2008]
  - Capacity at low-SNR
  - Upper and lower bounds on the capacity-achieving input
- Energy detection with equiprobable signaling
  - [Mallik, Murch; 2014]: “regular” SIMO systems
  - [Knott, Chowdhury, Manolakos, Goldsmith; 2014]: “massive” SIMO systems
- Very little is reported on the error performance with optimal detection

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System Model

- Single-input and single-output signal model:

$$y = hx + n$$

  - $h \sim \mathcal{CN}(0, \sigma_h^2)$
  - $n \sim \mathcal{CN}(0, \sigma_n^2)$
  - $x \in \mathcal{X} = \{x_1, \ldots, x_N\}$: signal set
  - $P_j = \mathrm{Prob}(x = x_j)$: prior probabilities
  - $\gamma_i = \gamma(x_i) = \sigma_h^2 |x_i|^2 / \sigma_n^2$: instantaneous received SNR
- Sufficient statistic: $z = |y|^2 / \sigma_n^2$

$$\begin{aligned} f_{z|x}(z|x) &= \frac{\exp\left(-\frac{z}{1+\gamma(x)}\right)}{1+\gamma(x)} \quad z \geq 0 \\ &\triangleq \mathcal{M}(z, x). \end{aligned}$$

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ML Detection Error Probability: SISO

- With $x = x_j$

$$P_{e,j} = \mathrm{Prob}\left(\bigcup_{i \neq j} \mathcal{B}_{i,j} \middle| x = x_j\right)$$
$$= \sum_{l=1}^{N-1} (-1)^{l-1} \sum_{\mathcal{I}_j \subset \mathcal{S}_j, |\mathcal{I}_j| = l} \mathrm{Prob}\left(\bigcap_{i \in \mathcal{I}_j} \mathcal{B}_{i,j} \, | x = x_j\right)$$

- $\mathcal{B}_{i,j}$: confusing x_i for x_j

$$\mathcal{B}_{i,j} \triangleq \{z \geq 0 : \mathcal{M}(z, x_i) > \mathcal{M}(z, x_j)\}$$
$$= \begin{cases} \left\{z : z > \frac{(1+\gamma_i)(1+\gamma_j)}{\gamma_i - \gamma_j} \log\left(\frac{1+\gamma_i}{1+\gamma_j}\right)\right\} & \text{if} \quad \gamma_i > \gamma_j \\ \left\{z : z < \frac{(1+\gamma_i)(1+\gamma_j)}{\gamma_j - \gamma_i} \log\left(\frac{1+\gamma_j}{1+\gamma_i}\right)\right\} & \text{if} \quad \gamma_i < \gamma_j \end{cases}$$

- Overall probability of error

$$P_{e,ML} = \sum_{j=1}^{N} P_j P_{e,j}$$

ML Detection Error Probability: SISO

- Since $z|x_i$ is an exponential rv with mean $1+\gamma_i$

$$
\begin{aligned}
&\text{Prob}\left(\mathcal{B}_{\mathcal{I}_j,j}\,|x=x_j\right)\\
&=\text{Prob}\left(\left\{\bigcap_{i\in\mathcal{I}_{j,A}}\{z:z>\lambda_{i,j}\}\right\}\bigcap\left\{\bigcap_{i\in\mathcal{I}_{j,B}}\{z:z<\lambda_{i,j}\}\right\}|x=x_j\right)\\
&=\left[e^{-\max_{i\in\mathcal{I}_{j,A}}\mu_{i,j}}-e^{-\min_{i\in\mathcal{I}_{j,B}}\mu_{i,j}}\right]\times 1_{\left\{\max_{i\in\mathcal{I}_{j,A}}\mu_{i,j}<\min_{i\in\mathcal{I}_{j,B}}\mu_{i,j}\right\}}
\end{aligned}
$$

- $\mathcal{I}_j$: A proper subset of $\mathcal{S}_j=\{1,\ldots,N\}\setminus\{j\}$

$$
\begin{aligned}
\mathcal{I}_{j,A} &\triangleq \left\{i\in\mathcal{I}_j:\gamma_i>\gamma_j\right\}\\
\mathcal{I}_{j,B} &\triangleq \left\{i\in\mathcal{I}_j:\gamma_i<\gamma_j\right\}\\
\lambda_{i,j} &\triangleq \frac{(1+\gamma_i)(1+\gamma_j)}{\gamma_i-\gamma_j}\log\left(\frac{1+\gamma_i}{1+\gamma_j}\right)\\
\text{and}\quad \mu_{i,j} &\triangleq \frac{\lambda_{i,j}}{1+\gamma_j}
\end{aligned}
\tag{1}
$$

ML Detection Error Probability: SIMO

- L: Number of receive antennas
- Straightforward extension since z is now a conditional Gamma rv
- Confusion event $\mathcal{B}_{i,j}$ is also simple in form

$$\mathcal{B}_{i,j} = \begin{cases} \left\{z : z > L\lambda_{i,j}\right\} & \text{if} \quad \gamma_i > \gamma_j \\ \left\{z : z < L\lambda_{i,j}\right\} & \text{if} \quad \gamma_i < \gamma_j \end{cases}$$

- Closed-form expression for $\mathrm{Prob}\left(\mathcal{B}_{\mathcal{I}_j,j} \,\middle|\, x = x_j\right)$ in terms of incomplete Gamma function
- Relies on i.i.d fading across diversity branches

Extensions: Coded Sequence Detection

- A SIMO system with i.i.d fading in space and time
- P: Codeword length
- Confusion event, $\mathcal{B}_{i,j}$: hyperplane in P dimensional i.i.d Gamma rvs
- $\mathcal{B}_{\mathcal{I}_j,j} = \bigcap_{i \in \mathcal{I}_j} \mathcal{B}_{i,j}$: intersection of $|\mathcal{I}_j|$ hyperplanes
- $\mathrm{Prob}\left(\mathcal{B}_{\mathcal{I}_j,j} \middle| x = x_j\right)$: A multi-dimensional integral
  - Can be evaluated using inverse Laplace transform or saddle-point integration techniques

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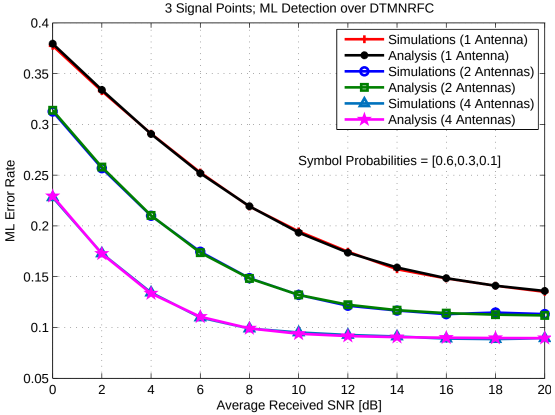
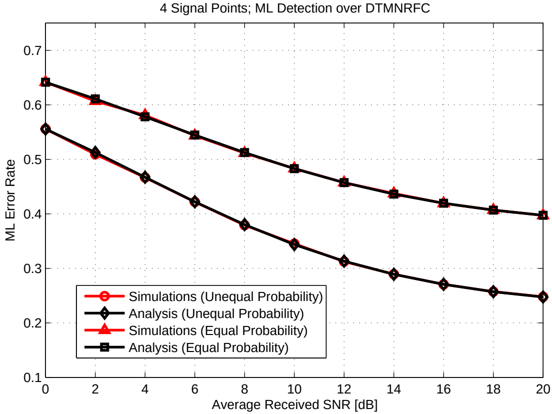
Average Probability of Error



3 signal points.

$P_1 = 0.6, P_2 = 0.3, P_3 = 0.1.$ $\gamma_1 = 0, \gamma_2 = 2$ and $\gamma_3 = 4.$

Average Probability of Error



4 signal points.

$P_1 = 0.4$, $P_2 = 0.4$, $P_3 = 0.1$, and $P_4 = 0.1$. $\gamma_1 = 0$, $\gamma_2 = 2$, $\gamma_3 = 4$, and $\gamma_4 = 6$.

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Optimal Signal Constellation: Challenges

- Unknown number of mass points, N
- Unknown prior probabilities, $\{P_i\}$
- Unknown mass point locations, $\{|x_i|\}$
- Not clear whether $P_{e,ML}$ is convex in $(\{P_i\}, \{|x_i|\})$

Making Progress: Parametric Search

- Start with the information-theoretic result that one mass point is at origin
- Fix the number of constellation points
- A geometric sequence of prior probabilities:

$$P_i = P_0 \alpha^{i-1}, \quad i = 1, \ldots, N$$

- A geometric sequence of mass points:

$$\begin{aligned} \gamma_i &= \beta^{i-1}, \quad i = 2, \ldots, N \\ \gamma_1 &= 0 \end{aligned}$$

- Constraints:

$$\sum_{i=1}^{N} P_i = 1 \implies P_0 = \frac{1-\alpha}{1-\alpha^N} \quad 0 < \alpha < 1$$

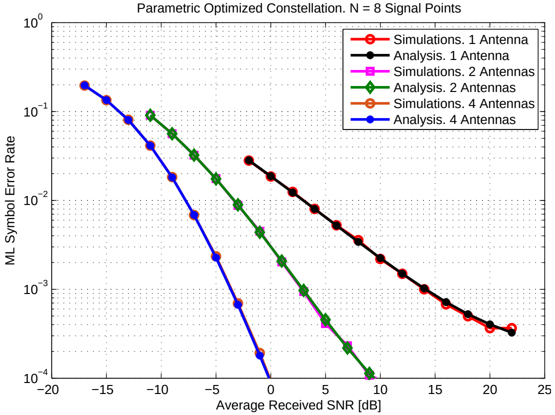
$$\sum_{i=1}^{N} P_i \gamma_i = \overline{\gamma} \implies \sum_{i=2}^{N} (\beta\alpha)^i - \overline{\gamma}/P_0 = 0$$

Optimal Constellations: Preliminary Results

With $L = 1$ receive antennas:

SNR (dB)	N	$P_{e,ML,\min}$	α_{opt}	β_{opt}
0	8	1.87×10^{-2}	0.01	50.4604
5	8	1.22×10^{-2}	0.01	80.2650
10	8	9.1×10^{-3}	0.01	109.1809
20	8	5.9×10^{-3}	0.01	171.1322

Parametric Constellation Optimization



8 signal points. $N_R = 1, 2$ and 4 receive antennas.

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Concluding Remarks

Take-away Message

1. Analyzed the ML error probability of DTM-NRFC
2. Uncoded and coded transmission with multiple receive antennas
3. Surprisingly simple closed-form error probability expression for the uncoded case
4. Useful tool to validate the relative performances of two signal sets
5. Too many unknowns for practical constellation optimization
6. Parametric approach lead to a manageable search space

Outlook

1. Analysis of MAP detection on DTM-NRFC (we made good progress ☺)
2. Noncoherent joint-source coding
3. Simplified massive SIMO systems
4. Improved constellation designs