Srinivasa Ramanujan and Signal Processing

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1887 - 1920

Self-educated Indian mathematician

Grew up in poverty (Kumbakonam, Tamil Nadu)

His genius discovered by Prof. G. H. Hardy

Worked with Hardy in 1914 - 1919 (Cambridge)

1877 - 1947

1887 -- 1920

Ramanujan created history in mathematics

Became a Fellow of the Royal Society at 32

Passed away at 33

A SYNOPSIS

OF

ELEMENTARY RESULTS

IN

PURE AND APPLIED MATHEMATICS:

CONTAINING

PROPOSITIONS, FORMULÆ, AND METHODS OF ANALYSIS,
WITH
ABRIDGED DEMONSTRATIONS.

BY

G. S. CARR, B.A.,

LATE PRIZEMAN AND SCHOLAR, OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

Expansion of the sine and cosine in factors.

807 $$x^{2n}-2x^ny^n\cos n\theta+y^{2n}$$
$$=\left\{x^2-2xy\cos\theta+y^2\right\}\left\{x^2-2xy\cos\left(\theta+\frac{2\pi}{n}\right)+y^2\right\}\ldots$$

to n factors, adding $\frac{2\pi}{n}$ to the angle successively.

Proof.—By solving the quadratic on the left, we get $x=y(\cos n\theta+i\sin n\theta)^{\frac{1}{n}}$. The n values of x are found by (757) and (626), and thence the factors. For the factors of $x^n\pm y^n$ see (480).

808 $$\sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\ldots$$

as far as n factors of sines.

Proof.—By putting $x=y=1$ and $\theta=2\phi$ in the last.

809 If n be even,
$$\sin n\phi=2^{n-1}\sin\phi\cos\phi\left(\sin^2\frac{\pi}{n}-\sin^2\phi\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\phi\right)\&c.$$

810 If n be odd, omit $\cos\phi$ and make up n factors, reckoning two factors for each pair of terms in brackets.

Obtained from (808), by collecting equidistant factors in pairs, and applying (659).

811 $$\cos n\phi=2^{n-1}\sin\left(\phi+\frac{\pi}{2n}\right)\sin\left(\phi+\frac{3\pi}{2n}\right)\ldots\text{ to }n\text{ factors.}$$

Proof.—Put $\phi+\frac{\pi}{2n}$ for ϕ in (808).

812 Also, if n be odd,
$$\cos n\phi=2^{n-1}\cos\phi\left(\sin^2\frac{\pi}{2n}-\sin^2\phi\right)\left(\sin^2\frac{3\pi}{2n}-\sin^2\phi\right)\ldots$$

813 If n be even, omit $\cos\phi$.

Proved as in (809).

814 $$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}.$$

Proof.—Divide (809) by $\sin\phi$, and make ϕ vanish; then apply (754).

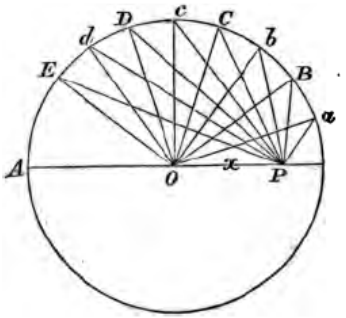
815 $$\sin\theta=\theta\left\{1-\left(\frac{\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{2\pi}\right)^2\right\}\left\{1-\left(\frac{\theta}{3\pi}\right)^2\right\}\ldots\ldots$$

816 $$\cos\theta=\left\{1-\left(\frac{2\theta}{\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{3\pi}\right)^2\right\}\left\{1-\left(\frac{2\theta}{5\pi}\right)^2\right\}\ldots\ldots$$

Proof.—Put $\phi=\frac{\theta}{n}$ in (809) and (812); divide by (814) and make n infinite.

817 $$e^x-2\cos\theta+e^{-x}$$
$$=4\sin^2\frac{\theta}{2}\left\{1+\frac{x^2}{\theta^2}\right\}\left\{1+\frac{x^2}{(2\pi\pm\theta)^2}\right\}\left\{1+\frac{x^2}{(4\pi\pm\theta)^2}\right\}\ldots$$

Proved by substituting $x=1+\frac{z}{2n}$, $y=1-\frac{z}{2n}$, and $\frac{\theta}{n}$ for θ in (807), making n infinite and reducing one series of factors to $4\sin^2\frac{\theta}{2}$ by putting $z=0$.



De Moivre's Property of the Circle. — Take P any point, and $POB=\theta$ any angle,

$$BOC=COD=\&c.=\frac{2\pi}{n};$$
$$OP=x;\quad OB=r.$$

819 $$x^{2n}-2x^nr^n\cos n\theta+r^{2n}$$
$$=PB^2\,PC^2\,PD^2\ldots\text{ to }n\text{ factors.}$$

By (807) and (702), since $PB^2=x^2-2xr\cos\theta+r^2$, &c.

820 If $x=r$, $2r^n\sin\frac{n\theta}{2}=PB\,.\,PC\,.\,PD\ldots\&c.$

821 *Cotes's properties.*—If $\theta=\frac{2\pi}{n}$,
$$x^n\sim r^n=PB\,.\,PC\,.\,PD\ldots\&c.$$

822 $$x^n+r^n=Pa\,.\,Pb\,.\,Pc\ldots\&c.$$

ADDITIONAL FORMULÆ.

823 $\cot A+\tan A=2\operatorname{cosec}2A=\sec A\operatorname{cosec}A.$

824 $\operatorname{cosec}2A+\cot 2A=\cot A.\quad \sec A=1+\tan A\tan\frac{A}{2}.$

826 $$\cos A=\cos^4\frac{A}{2}-\sin^4\frac{A}{2}.$$

827 $$\tan A+\sec A=\tan\left(45^\circ+\frac{A}{2}\right).$$

828 $$\frac{\tan A+\tan B}{\cot A+\cot B}=\tan A\tan B.$$

829. $$\sec^2A\operatorname{cosec}^2A=\sec^2A+\operatorname{cosec}^2A.$$

Ramanujan’s slate

vi $\phi(x) + \phi(-x) = 2\phi(x^4)$

vii $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

viii $\phi(x)\,\phi(-x) = \phi^2(-x^2)$.

ix $\phi(x)\,\psi(x^2) = \psi^2(x)$

x. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

xi. $\phi^2(x) + \phi^2(-x) = 2\,\phi^2(x^2)$

xii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$.

xiii. $\psi^2(x) + \psi^2(-x) = 2\,\psi(x^2)\,\phi(x^4)$

xiv. If $\left(\frac{1-z}{1+z}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$ then $1-z^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$

Ex. 1. $\frac{\psi(x)}{\psi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$

2. $\psi(x)\,\psi(-x) = \psi(x^2)\,\phi(-x^2)$

3. $\frac{\psi(x)\,\psi(-x)}{\psi(x^2)\,\psi(-x^2)} = \frac{\psi(-x^2)}{\psi(x^4)}$.

18. If $F(x) = e^{-\pi\cdot\frac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\left(\frac{1.3.5}{2.4.6}\right)^2(1-x)^3+\&c}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\left(\frac{1.3.5}{2.4.6}\right)^2x^3+\&c}}$

i. $F(x) = \frac{x}{16}\,e^{4\cdot\frac{\left(\frac{1}{2}\right)^2\frac{1}{1.2}x+\left(\frac{1.3}{2.4}\right)^2\left(\frac{1}{1.2}+\frac{1}{3.4}\right)x^2}{1+\left(\frac{1}{2}\right)^2x+\left(\frac{1.3}{2.4}\right)^2x^2+\&c}}$

ii. $F\left(1-\frac{1}{x}\right) + \theta = \frac{\log_e x}{10+\sqrt{36+(\log_e x)^2}}$ ~~where θ is numerically much less than~~ ~~$\frac{2}{135}F^5\left(1-\frac{1}{x}\right)$~~. $\theta = \frac{1}{2160}\cdot\left\{\frac{\log_e x}{8+(\log_e x)^2}\right\}^5$.

iii. $\log_e F(x)\,\log_e F(1-x) = \pi^2$

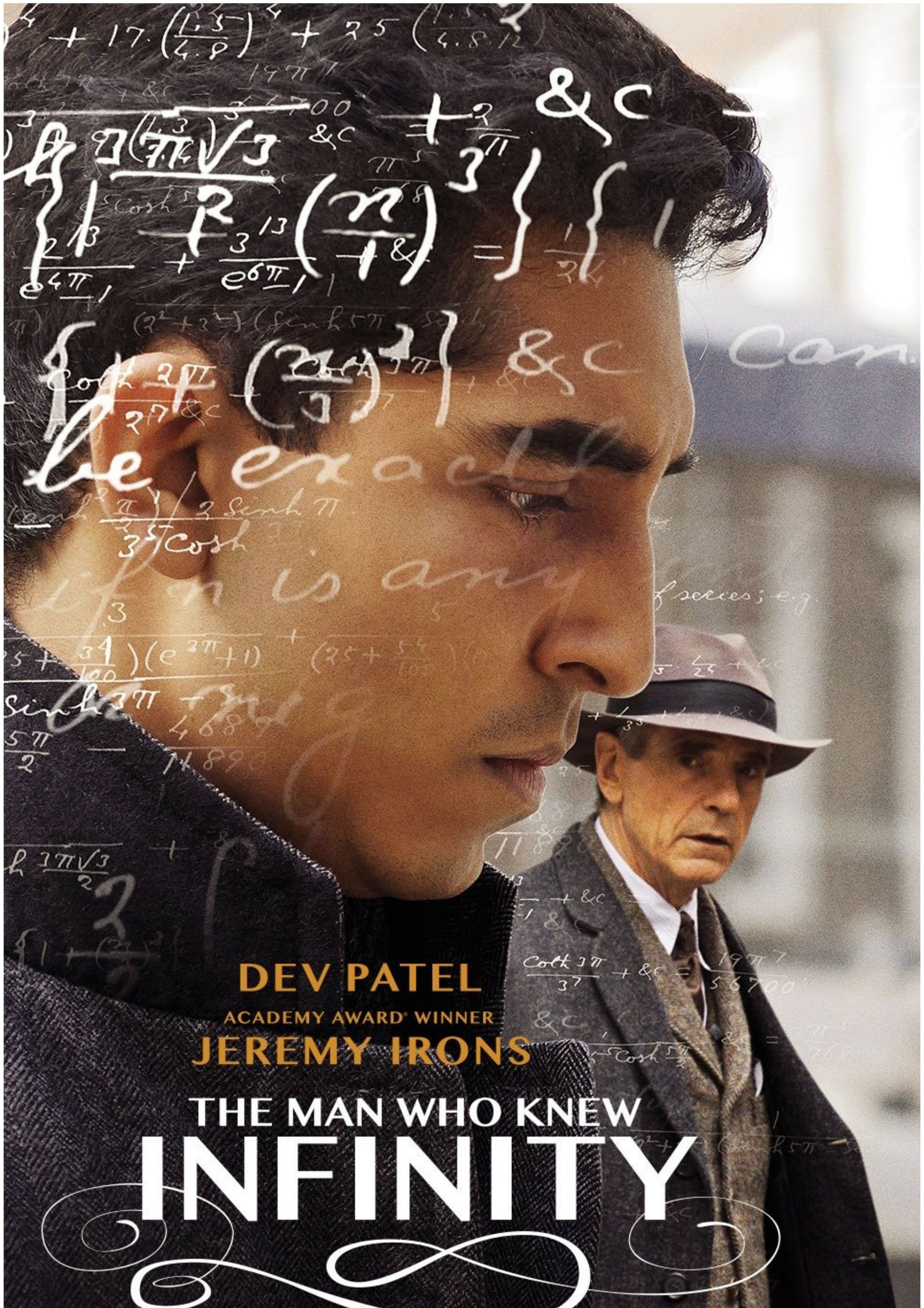
iv $F(1-x) + F\left(1-\frac{1}{x}\right) = 0$

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^2)}$

N.B. If we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms, then we can find it's expansion to 2n terms as follows —

Suppose we know the expansion of $F\left(\frac{2x}{1+x}\right)$ to n terms.

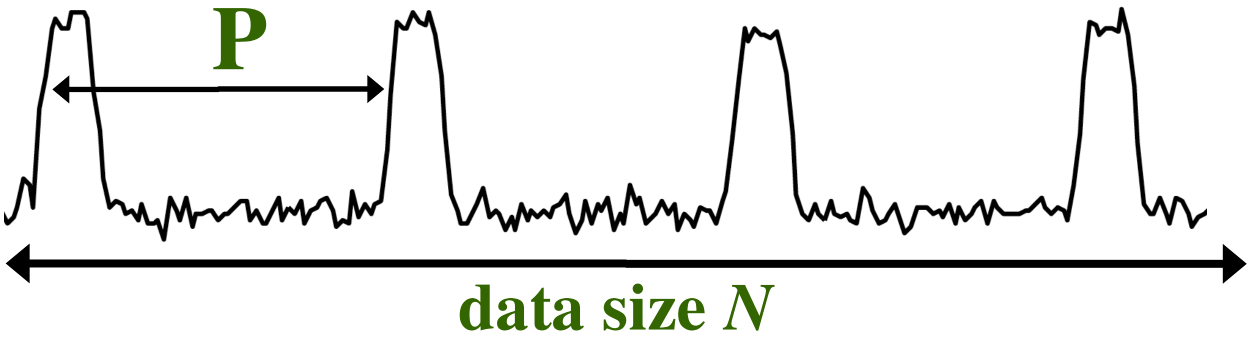

MM


DEV PATEL
ACADEMY AWARD® WINNER
JEREMY IRONS
THE MAN WHO KNEW
INFINITY

Talk Outline

- **Ramanujan sums (RS): 1918**
- **Representing periodic signals**
- **From RS to Subspaces**
- **From Subspaces to Dictionaries**
- **From Dictionaries to Filter Banks**
- **iMUSIC**
- **Conclusions, Acknowledgements, …**

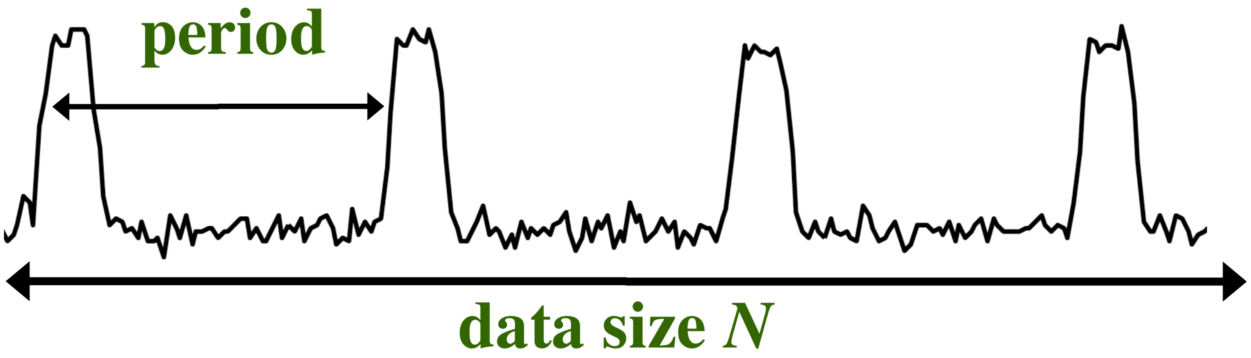
Periodic $x(n)$



$$x(n) = x(n + P)$$

Smallest such integer P is called the period

Periodic $x(n)$



DFT representation:

$$x(n) = \sum_{k=0}^{N-1} X[k]\boxed{e^{j(2\pi k/N)n}}$$

period
= a divisor of N

Let $N = 6$, look at these

period 1 $e^{\frac{j2\pi(0n)}{6}}$

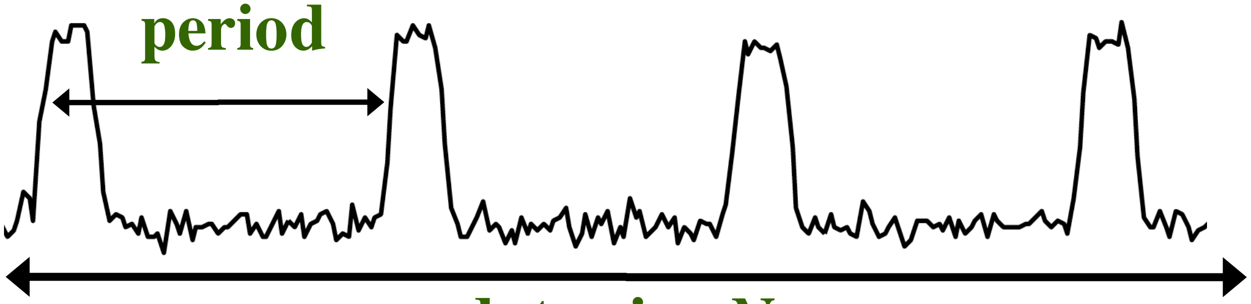
period 2 $e^{\frac{j2\pi(3n)}{6}}$

period 3 $e^{\frac{j2\pi(2n)}{6}}, \ e^{\frac{j2\pi(4n)}{6}}$

period 6 $e^{\frac{j2\pi n}{6}}, \ e^{\frac{j2\pi(5n)}{6}}$

periods 4 and 5 missing!

Periodic $x(n)$



DFT representation:

$$x(n) = \sum_{k=0}^{N-1} X[k]\boxed{e^{j(2\pi k/N)n}} \longleftarrow$$

period = N

or a divisor of N

N = 32; divisors = 1, 2, 4, 8, 16, 32

Very few periods in basis

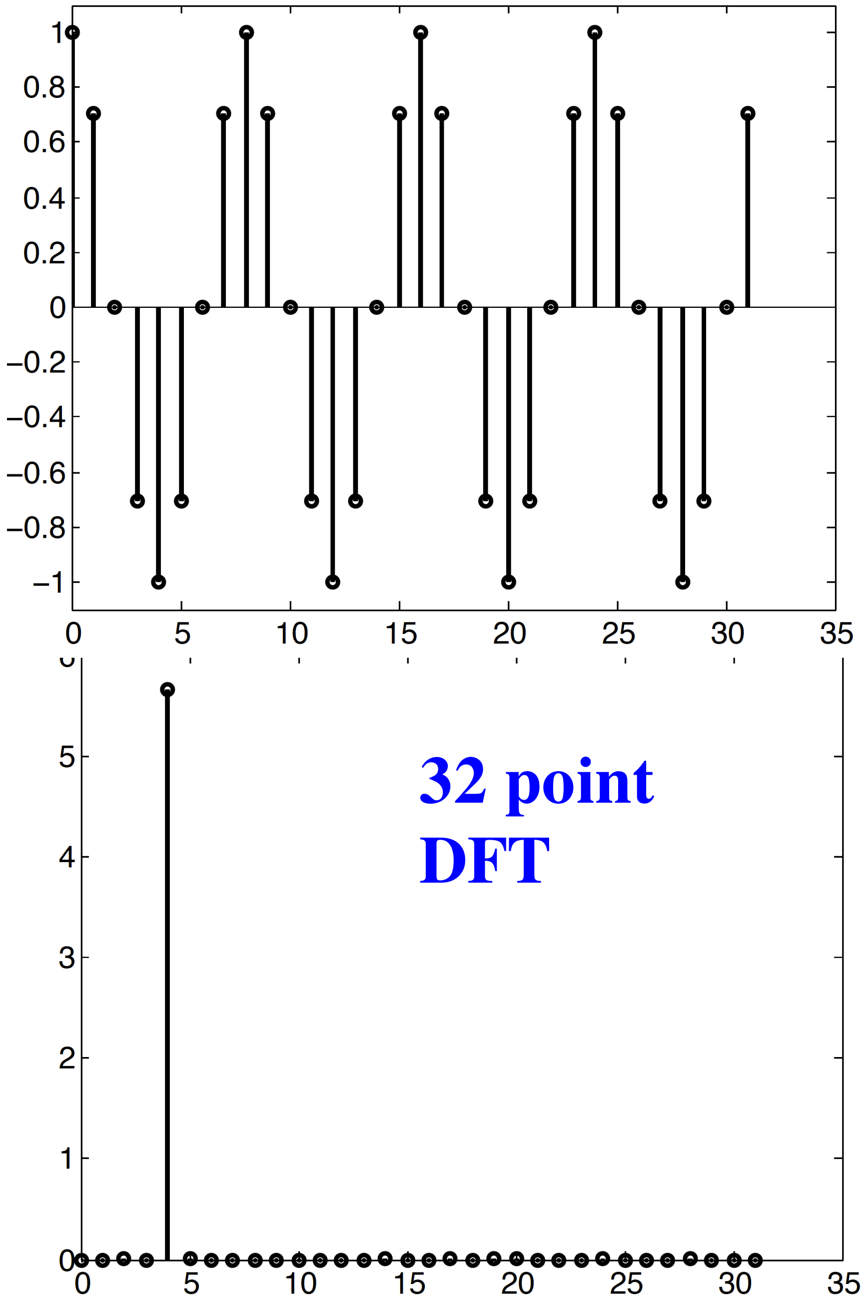
Ramanujan-sum representation: $x(n) = \sum_{q=1}^{N} a_q \underset{\uparrow}{c_q}(n)$

period q

Every period q is in basis!

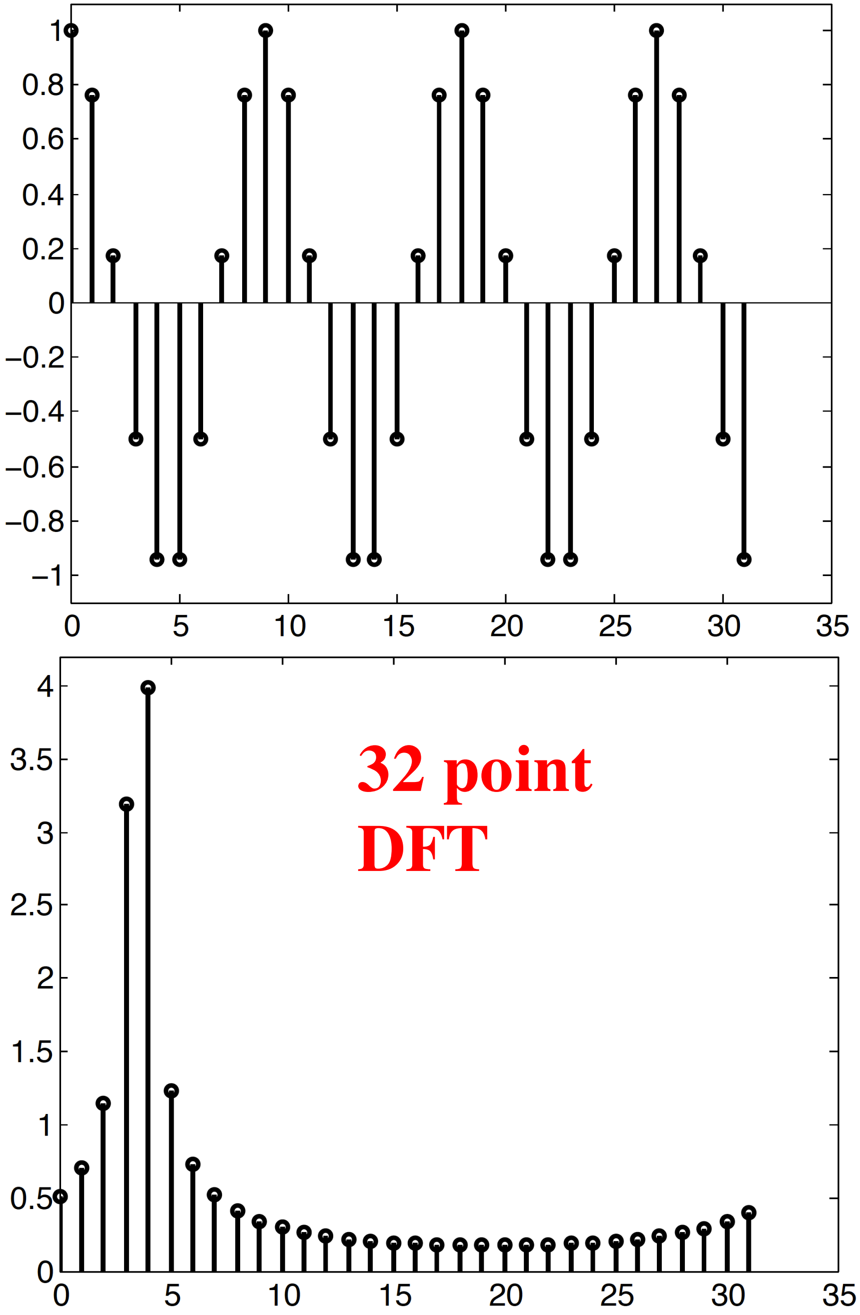
Limitations of DFT: Example

$e^{j(2\pi/8)n}$ **(Re part)**



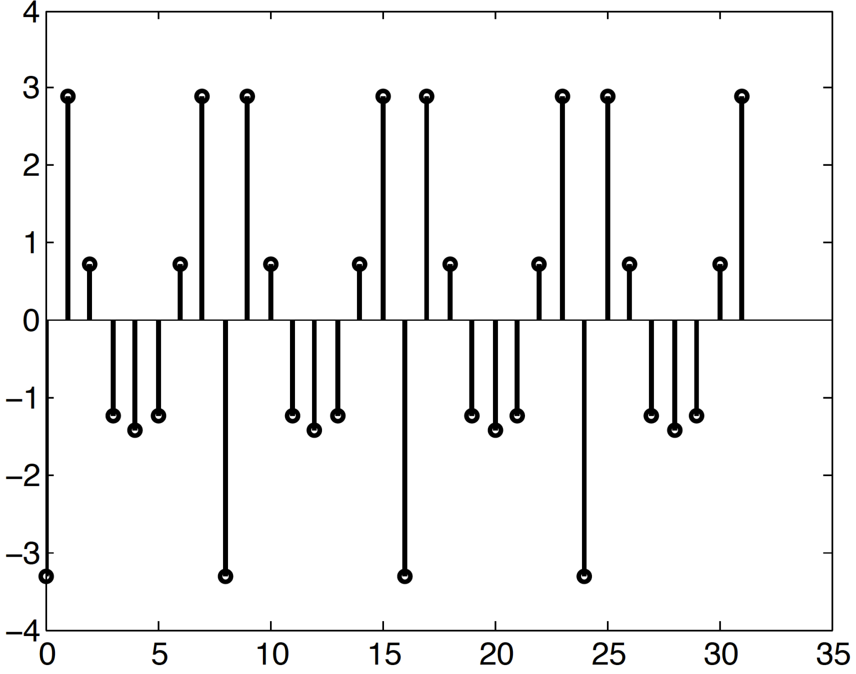
32 point DFT

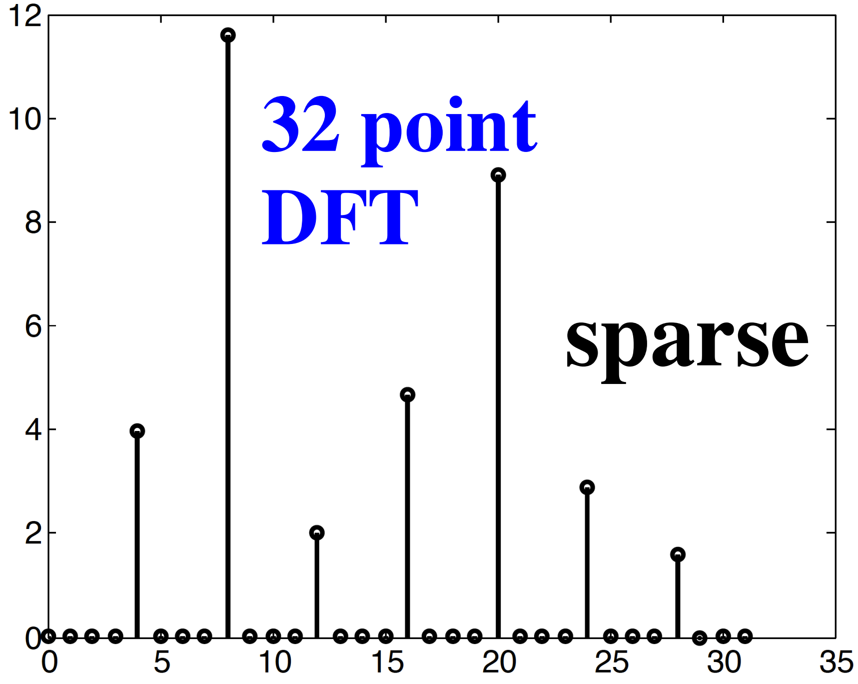
$e^{j(2\pi/9)n}$ **(Re part)**



32 point DFT

Period 8 (Re part)

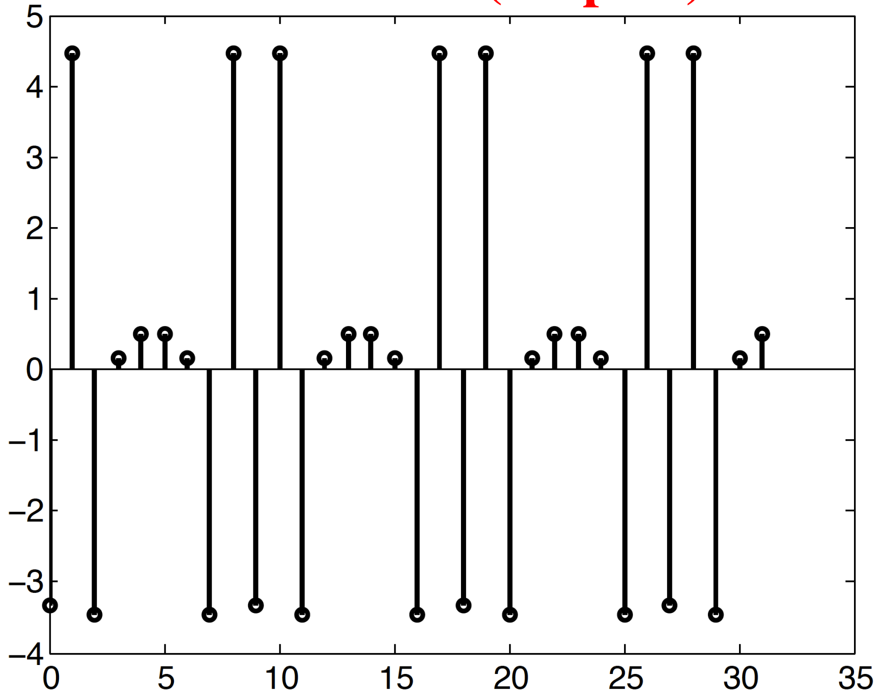


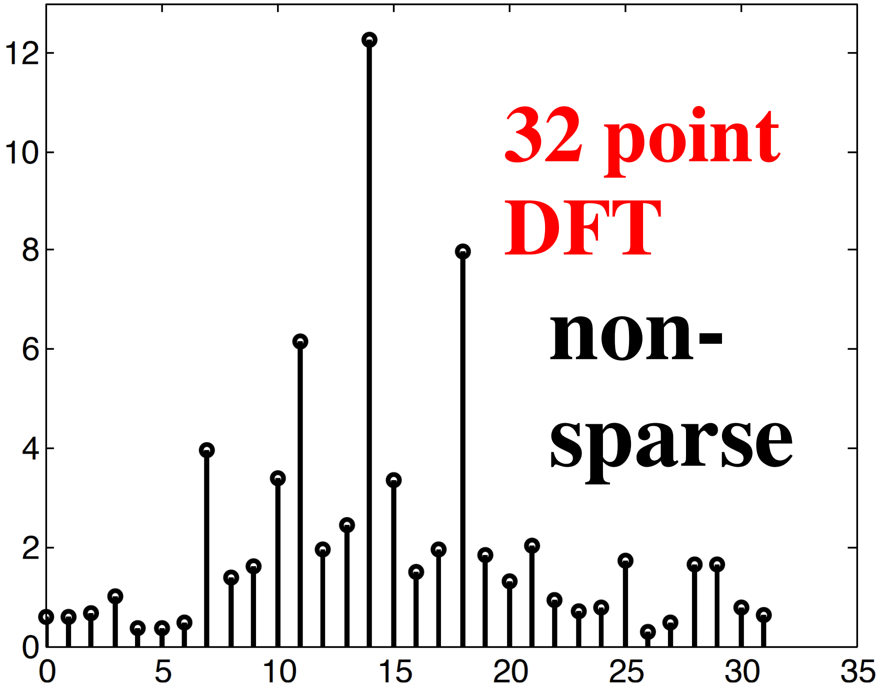


32 point DFT

sparse

Period 9 (Re part)

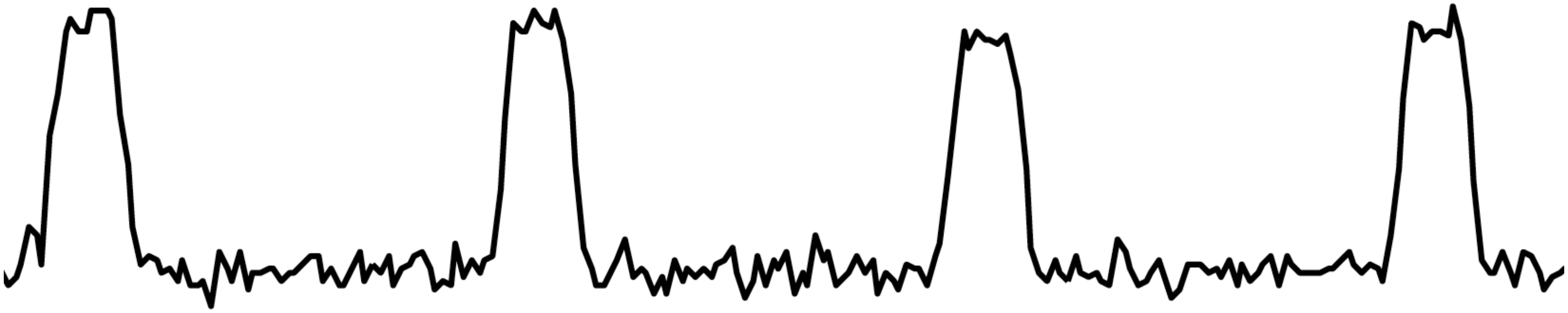


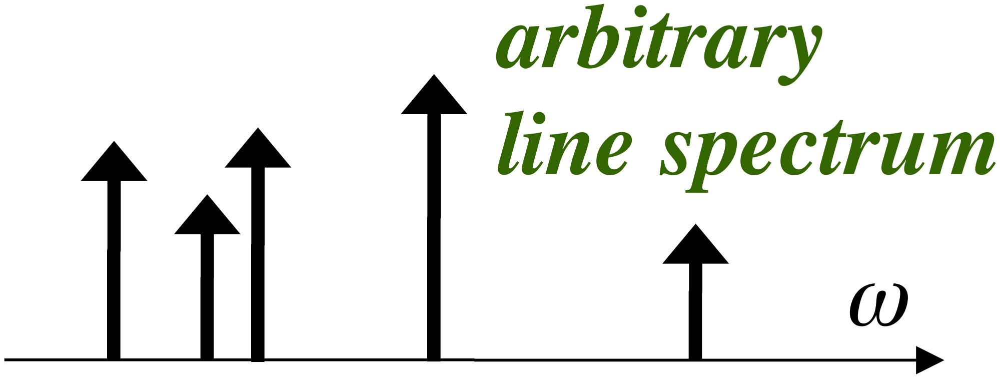


32 point DFT

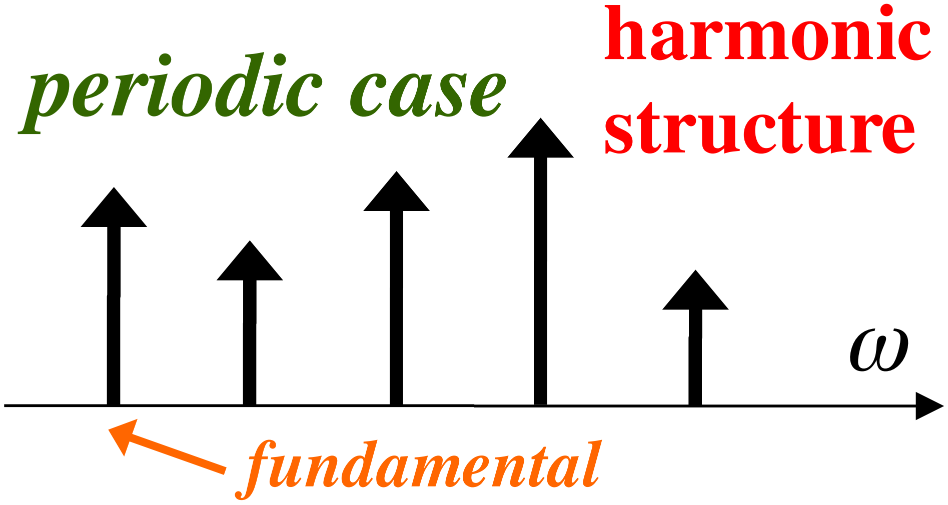
non-sparse

Identifying periods vs spectrum est.





arbitrary line spectrum



periodic case

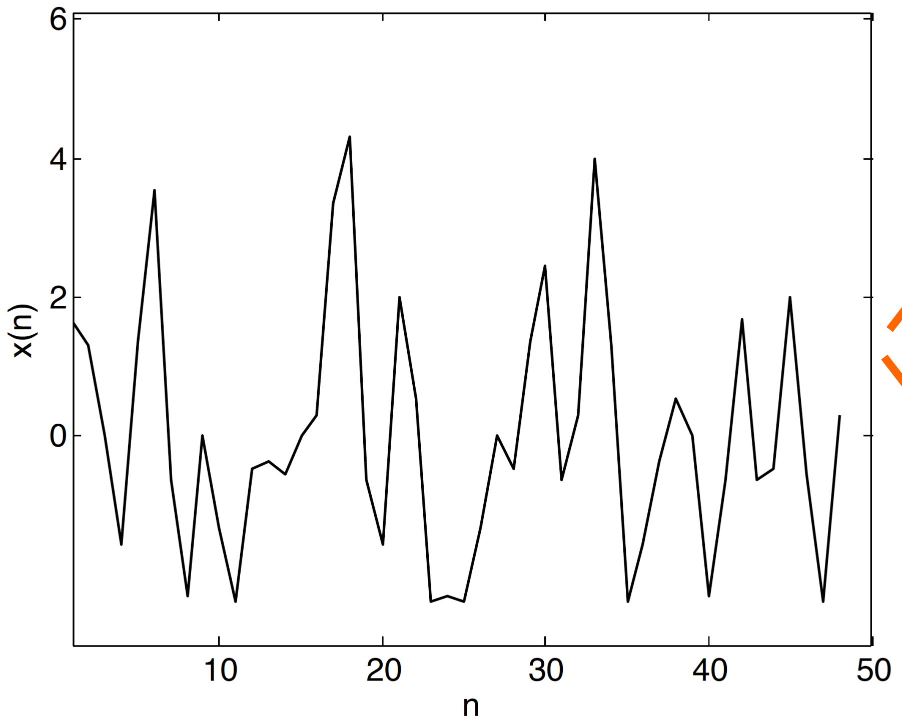
harmonic structure

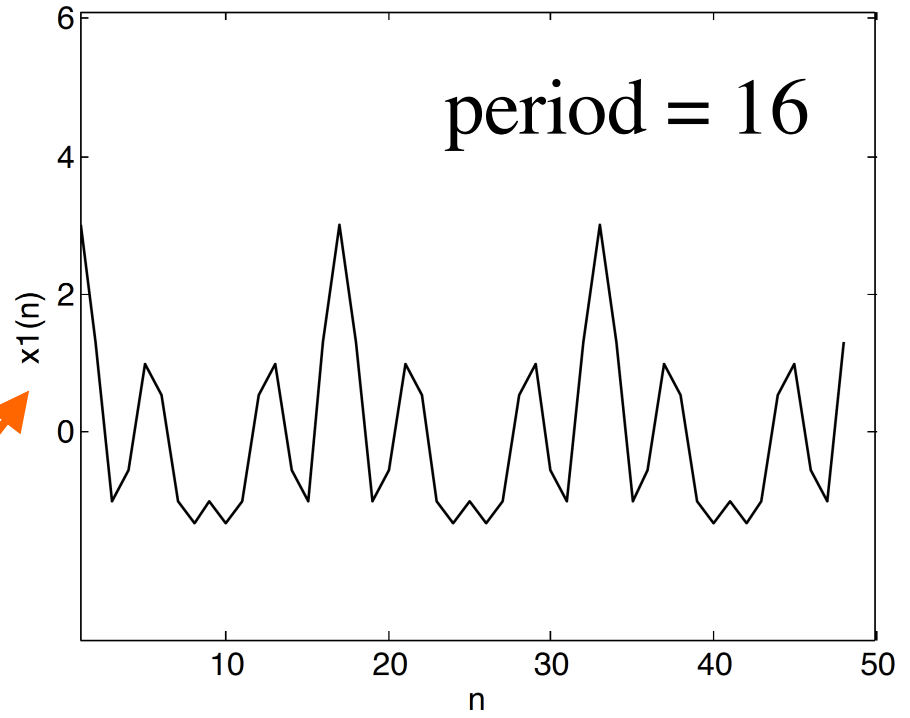
fundamental

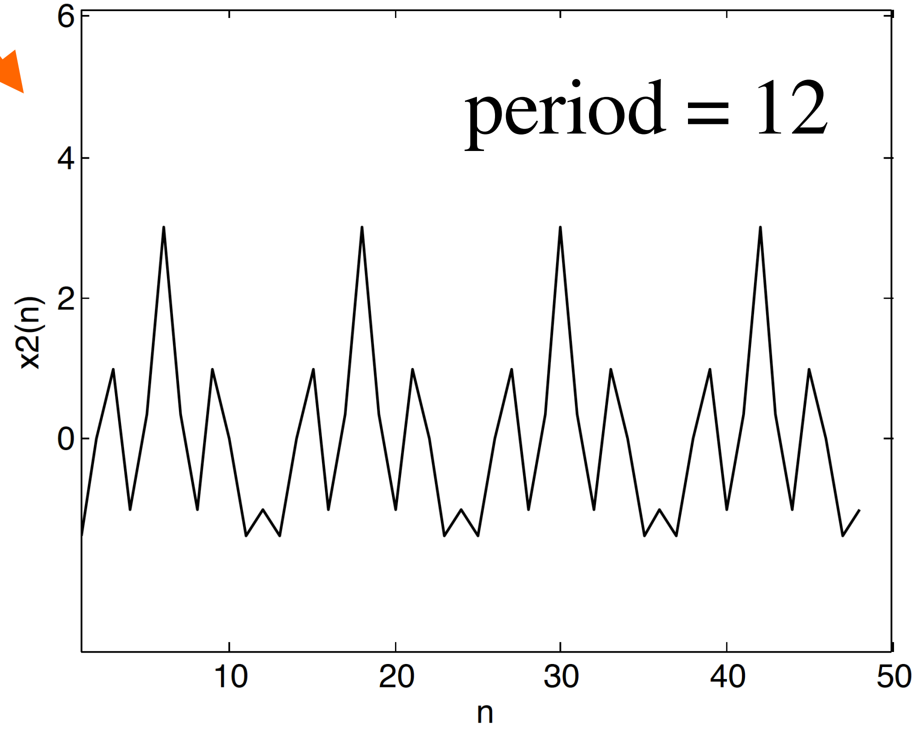
*DFT, MUSIC, **H**MUSIC, **H**MP, etc., are not the best …*

Ramanujan offers something new

Hidden periodic components



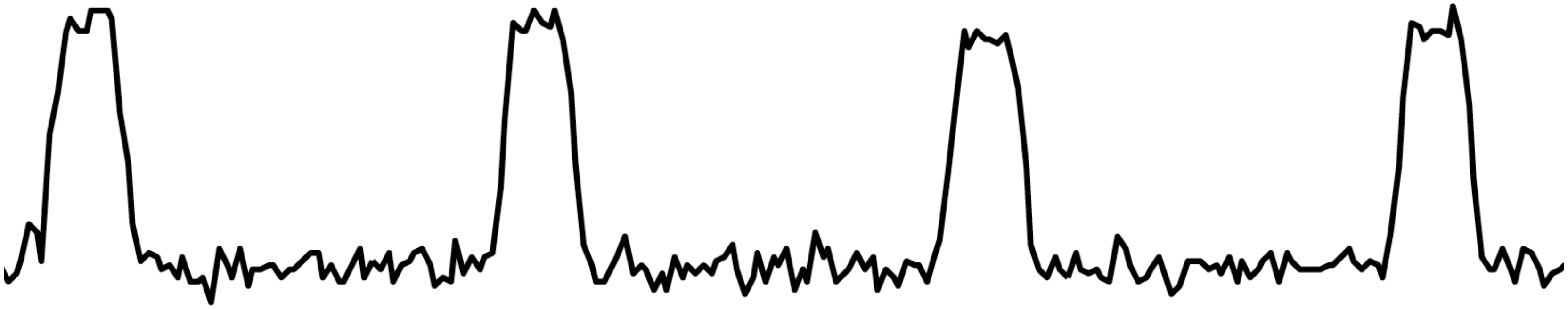




Does not "look" periodic

Ramanujan offers sparse representation …

Importance of periodicity



- Pitch identification acoustics (music, speech, …)
- Time delay estimation in sensor arrays
- Medical applications
- Genomics and proteomics
- Radar
- Astronomy
- Physics

Ramanujan sum (1918)

$$\boxed{c_q(n) = \sum_{\substack{k=1\\(k,q)=1}}^{q} e^{j2\pi kn/q}}$$

***q* = positive integer**

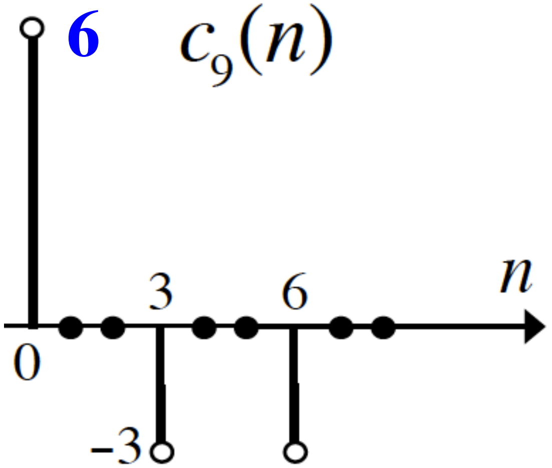
k and q coprime

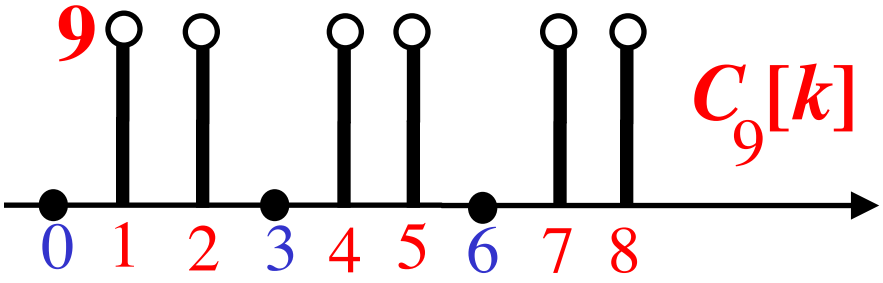
$$c_q(n+q) = c_q(n)$$

period *q*

# of terms = $\phi(q)$ = Euler totient

$$C_q[k] = \begin{cases} q & \text{if } (k,q)=1 \\ 0 & \text{otherwise.} \end{cases}$$





$$\boxed{c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}}$$

primitive frequencies with same period q

Theorem: Ramanujan sum is *integer* valued!

Examples:

$$\begin{aligned} c_1(n) &= 1 \\ c_2(n) &= 1, -1 \\ c_3(n) &= 2, -1, -1 \\ c_4(n) &= 2, 0, -2, 0 \\ c_5(n) &= 4, -1, -1, -1, -1 \\ c_6(n) &= 2, 1, -1, -2, -1, 1 \end{aligned}$$

Orthogonal: $$\sum_{n=0}^{m-1} c_{q_1}(n) c_{q_2}(n) = 0, \quad q_1 \neq q_2.$$

$$m = \mathrm{lcm}(q_1, q_2)$$

What did Ramanujan do with these?

He expanded arithmetic functions (1918):

Number-of-divisors:
$$\sigma_0(n) = -\sum_{q=1}^{\infty} \frac{\ln q}{q} c_q(n)$$

Sum-of-divisors:
$$\sigma(n) = \frac{n\pi^2}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}$$

Euler-totient:
$$\phi(n) = \frac{6n}{\pi^2}\Bigg(c_1(n) - \frac{c_2(n)}{2^2-1} - \frac{c_3(n)}{3^2-1} - \frac{c_5(n)}{5^2-1} + \frac{c_6(n)}{(2^2-1)(3^2-1)} - \frac{c_7(n)}{7^2-1} + \frac{c_{10}(n)}{(2^2-1)(5^2-1)} - \frac{c_{11}(n)}{11^2-1} - \frac{c_{13}(n)}{13^2-1} + \frac{c_{14}(n)}{(2^2-1)(7^2-1)} + \dots \Bigg)$$

von Mangoldt function:
$$\Lambda(n) = \frac{n}{\phi(n)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n)$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for prime } p, \text{ with } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Our goal

$$\boxed{c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}}$$

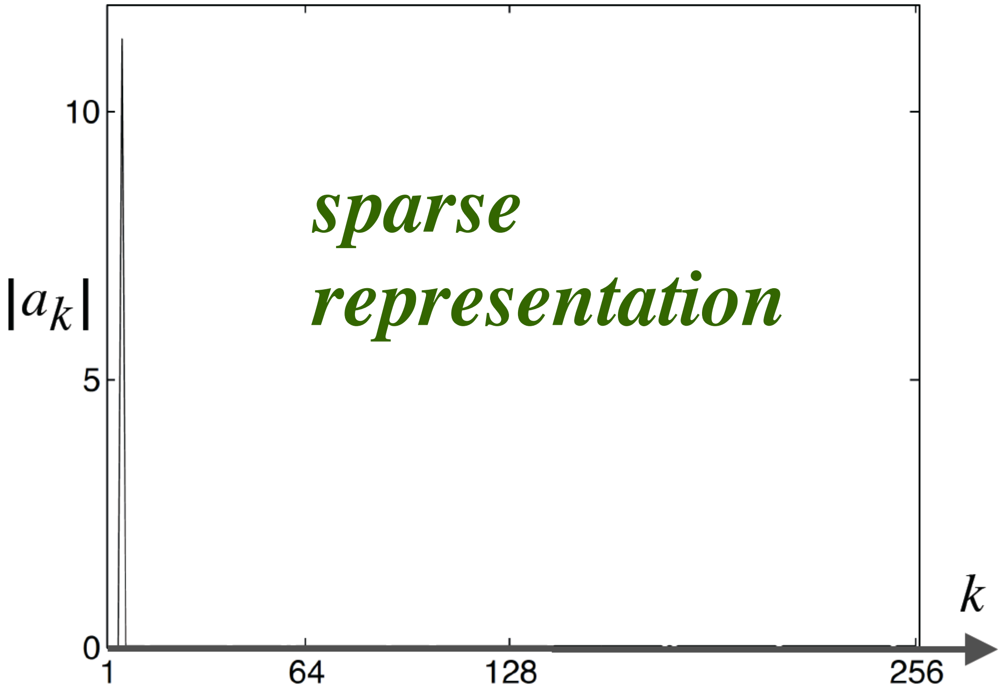
$$c_q(n+q) = c_q(n)$$

- **Use this to represent periodic signals efficiently**
- **Significant advantages over traditional …**

Representation for periodic signals?

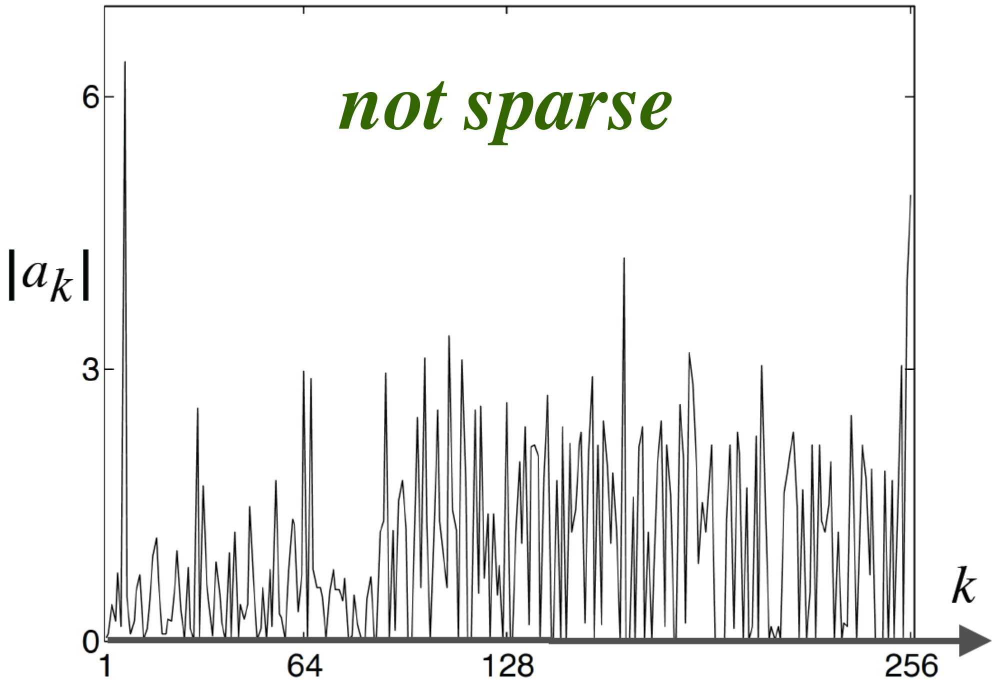
$$x(n) = \sum_{q=1}^{N} a_q c_q(n) \qquad c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{j2\pi kn/q}$$

$x(n) =$ $\cos(2\pi n/6)$



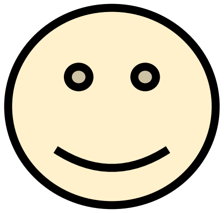
sparse representation

$x(n) =$ $\cos(2\pi n/7)$



not sparse

Not a good representation



What do we do about it?

$$x(n) = \sum_{q=1}^{N} a_q c_q(n)$$

Replace each Ramanujan-sum with a subspace:

Ramanujan subspace S_q

Leads to a nice representation! ☺

Ramanujan subspace S_q

[Vaidyanathan 2014, IEEE SP Trans.]

Space of signals of the form:

$$\sum_{\substack{k=1 \\ (k,q)=1}}^{q} a_k W_q^{kn}$$

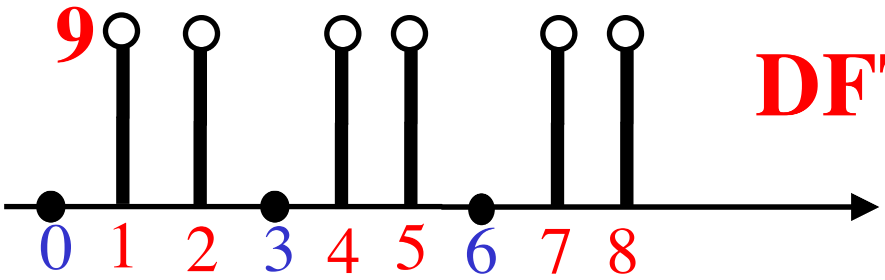
complex basis

$$= \sum_{l=0}^{\phi(q)-1} \beta_l c_q(n-l)$$

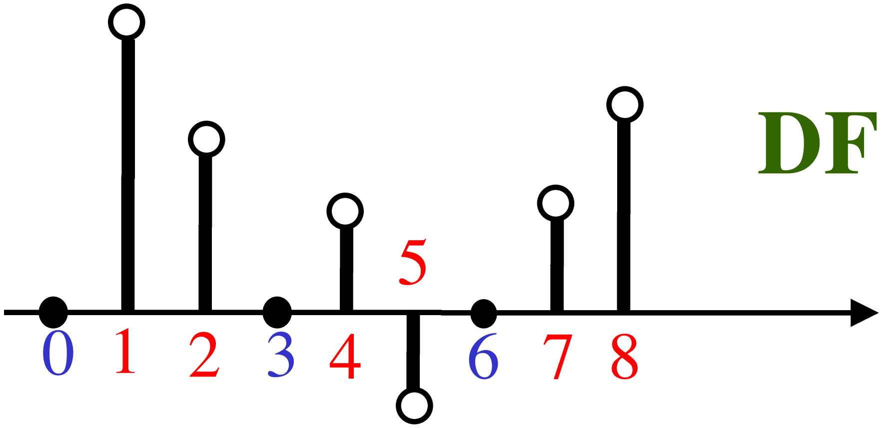
real integer basis

Dimension = $\phi(q)$ = Euler's totient function

Ramanujan subspace: look at the DFT

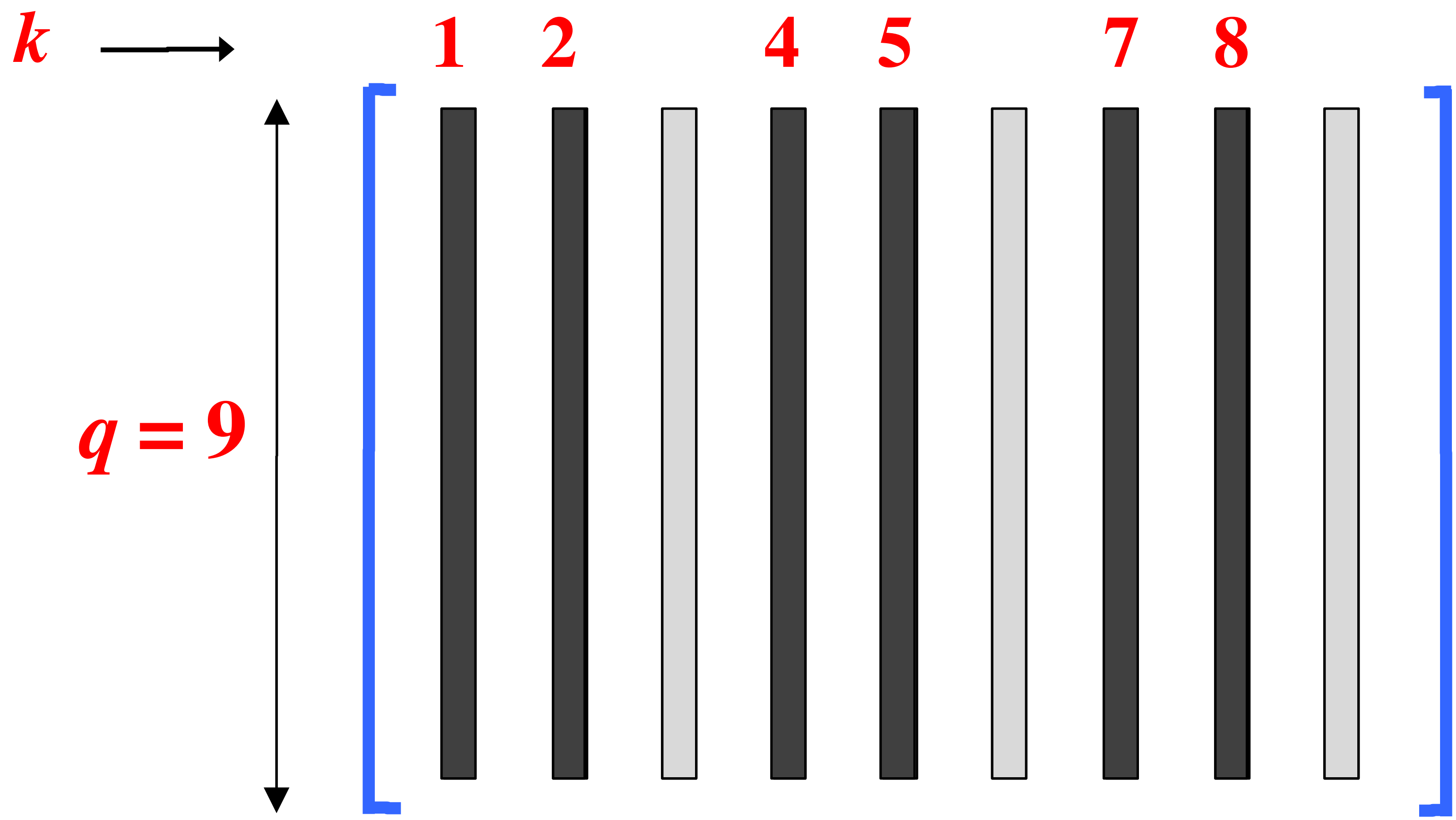


DFT of $c_9(n)$



DFT of a signal in $\mathcal{S}_9$

Think of the q x q DFT matrix



$c_q(n)$: *sum of dark cols.* **Ramanujan sum**

S_q : *space spanned by dark cols.* **Ramanujan subspace**

Periodicity Theorems [PPV 2014, IEEE SP Trans].

1. Nonzero signals in $\mathcal{S}_q$ have period *q*.

(can’t be smaller).

2. Any period-*P* signal can be written as

$$x(n) = \sum_{m=1}^{K} x_{q_m}(n) \qquad x_{q_m}(n) \in \mathcal{S}_{q_m}$$

where q_m are divisors of *P*.

Periodicity Theorems [PPV 2014, IEEE SP Trans].

3. Consider the sum $x(n) = \sum_{m=1}^{K} x_{q_m}(n)$

where $x_{q_m}(n) \in \mathcal{S}_{q_m}$**.**

This has period $= \mathrm{lcm}(q_1, q_2, \ldots, q_K)$

(can’t be smaller).

Farey dictionary (PPV and Piya Pal, 2014)

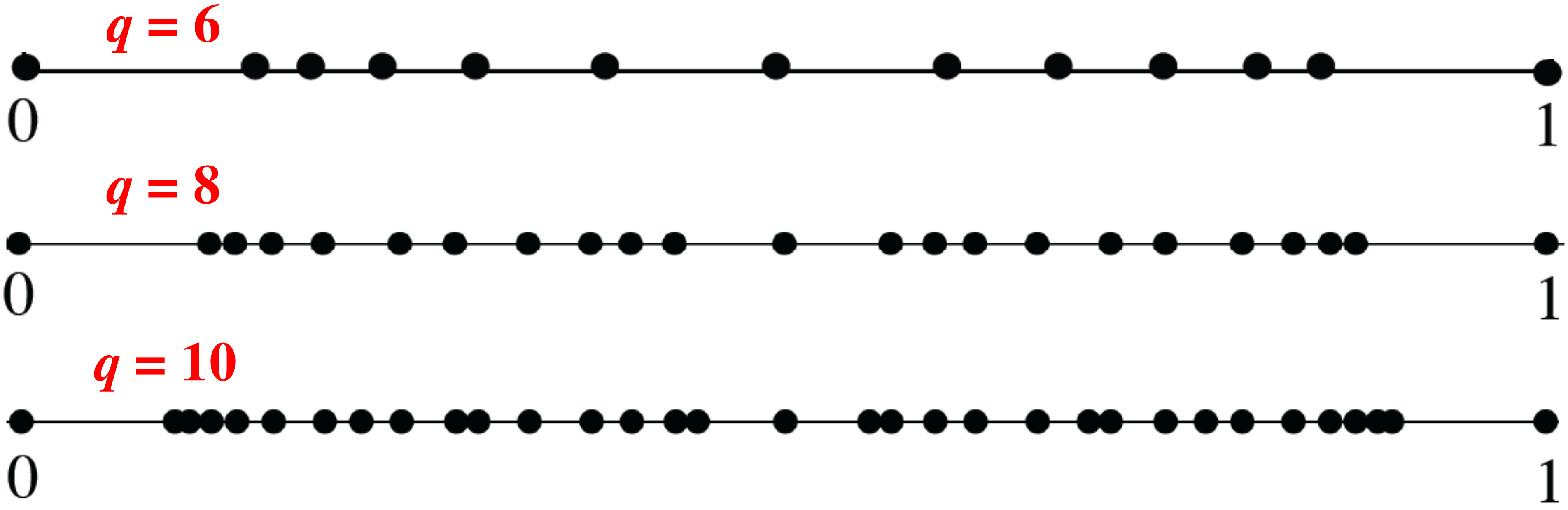
$$W_q = e^{-j2\pi/q}$$

$$\phi(4) = 2 \qquad \phi(5) = 4$$

$$\begin{array}{c|c|cc|cc|cccc|cc}
1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_2 & W_3 & W_3^2 & W_4 & W_4^3 & W_5 & W_5^2 & W_5^3 & W_5^4 & W_6 & W_6^5 \\
1 & W_2^2 & W_3^2 & W_3^4 & W_4^2 & W_4^6 & W_5^2 & W_5^4 & W_5^6 & W_5^8 & W_6^2 & W_6^{10} \\
1 & W_2^3 & W_3^3 & W_3^6 & W_4^3 & W_4^9 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} & W_6^3 & W_6^{15} \\
1 & W_2^4 & W_3^4 & W_3^8 & W_4^4 & W_4^{12} & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} & W_6^4 & W_6^{20} \\
1 & W_2^5 & W_3^5 & W_3^{10} & W_4^5 & W_4^{15} & W_5^5 & W_5^{10} & W_5^{15} & W_5^{20} & W_6^5 & W_6^{25} \\
\mathcal{S}_1 & \mathcal{S}_2 & \multicolumn{2}{c|}{\mathcal{S}_3} & \multicolumn{2}{c|}{\mathcal{S}_4} & \multicolumn{4}{c|}{\mathcal{S}_5} & \multicolumn{2}{c}{\mathcal{S}_6}
\end{array}$$

Farey frequency grid

Farey Frequency Grids



Non-uniform frequency grids for period estimation

Farey series, in *Number Theory* [Hardy and Wright 1938, 2008]

Ramanujan dictionary

[Srikanth Tenneti, PPV 2015, 2016, IEEE SP Trans].

$$\Phi(N) \triangleq \sum_{m=1}^{N} \phi(m) = \frac{3N^2}{\pi^2} + O(N \log N)$$

$$\begin{array}{cc}
 & \xleftrightarrow{\hspace{12em}} \\
\mathbf{A} = & \left[\begin{array}{c|c|cc|cc|cccc}
1 & 1 & 2 & -1 & 2 & 0 & 4 & -1 & -1 & -1 \\
1 & -1 & -1 & 2 & 0 & 2 & -1 & 4 & -1 & -1 \\
1 & 1 & -1 & -1 & -2 & 0 & -1 & -1 & 4 & -1 \\
1 & -1 & 2 & -1 & 0 & -2 & -1 & -1 & -1 & 4 \\
1 & 1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1
\end{array}\right] \Bigg\updownarrow N \\
 & \begin{array}{cccccccccc}
\mathcal{S}_1 & \mathcal{S}_2 & \multicolumn{2}{c}{\mathcal{S}_3} & \multicolumn{2}{c}{\mathcal{S}_4} & \multicolumn{4}{c}{\mathcal{S}_5}
\end{array}
\end{array}$$

$$\Phi(8) = 22, \Phi(10) = 32, \Phi(14) = 64, \Phi(32) = 324, \ldots$$

Frame, rather than basis

Finding period using Ramanujan dictionary

$$\mathbf{A} = \begin{array}{c|c|cc|cc|cccc|c} \phi(1) & \phi(2) & \multicolumn{2}{c|}{\phi(3)} & \multicolumn{2}{c|}{\phi(4)} & \multicolumn{4}{c|}{\phi(5)} & \\ \hline 1 & 1 & 2 & -1 & 2 & 0 & 4 & -1 & -1 & -1 & \uparrow \\ 1 & -1 & -1 & 2 & 0 & 2 & -1 & 4 & -1 & -1 & \\ 1 & 1 & -1 & -1 & -2 & 0 & -1 & -1 & 4 & -1 & N \\ 1 & -1 & 2 & -1 & 0 & -2 & -1 & -1 & -1 & 4 & \\ 1 & 1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & -1 & \downarrow \\ \hline \mathcal{S}_1 & \mathcal{S}_2 & \multicolumn{2}{c|}{\mathcal{S}_3} & \multicolumn{2}{c|}{\mathcal{S}_4} & \multicolumn{4}{c|}{\mathcal{S}_5} & \end{array}$$

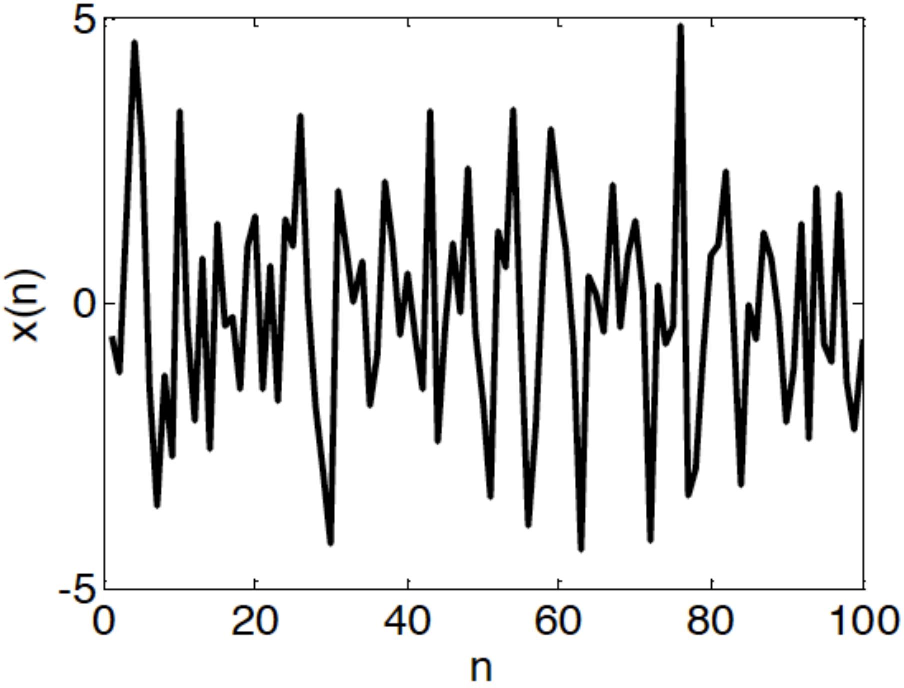
Given x, find a sparse representation: $\mathbf{x} = \mathbf{A}\mathbf{y}$

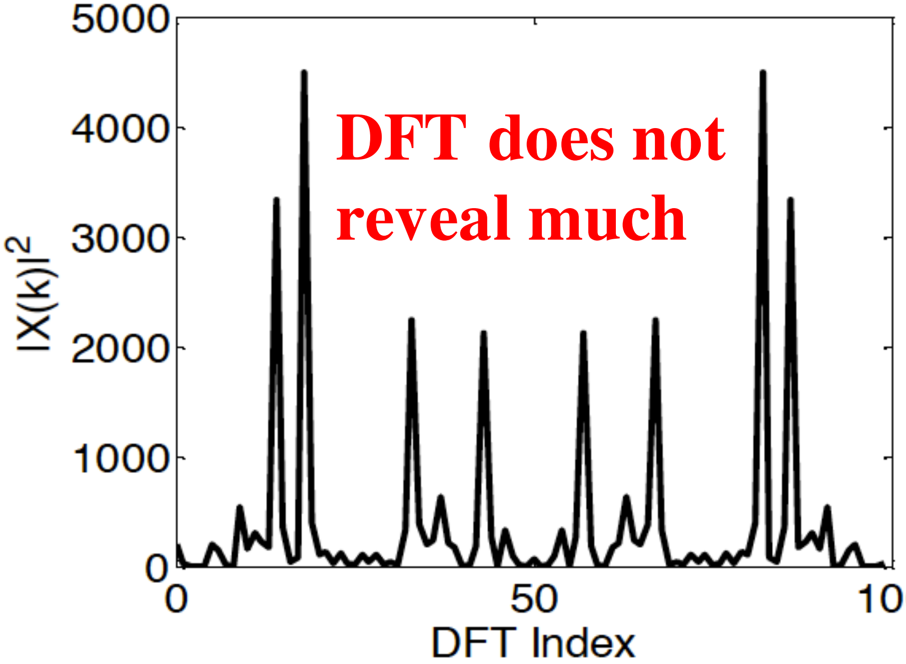
$$\boxed{x(n) = \sum_{m=1}^{K} x_{q_m}(n) \qquad x_{q_m}(n) \in \mathcal{S}_{q_m}}$$

Then period P $= \operatorname{lcm}(q_1, q_2, \ldots, q_K)$

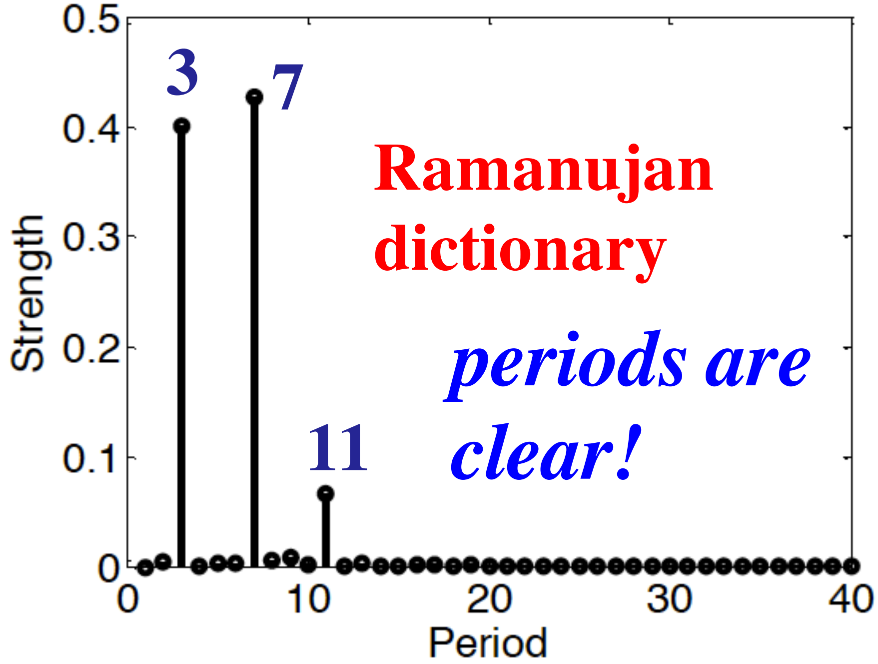
Example

Hidden periods: 3, 7, 11





DFT does not reveal much

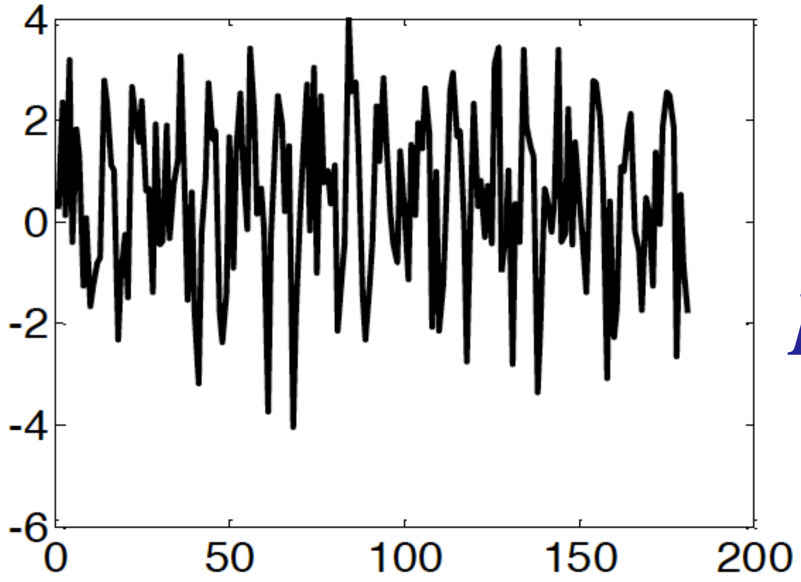


Ramanujan dictionary

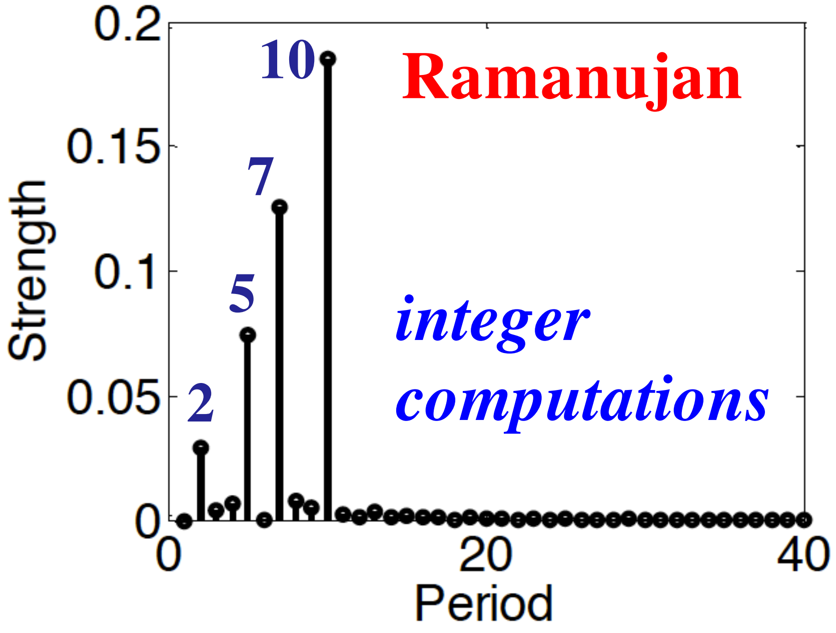
periods are clear!

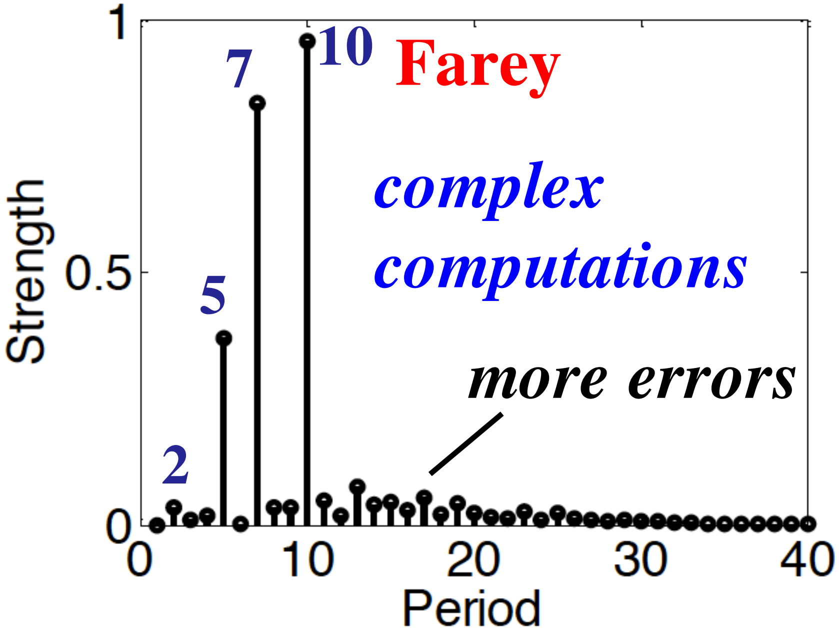
S. Tenneti, P. P. Vaidyanathan

Example



Hidden periods: 7, 10





Ramanujan vs other methods

Ramanujan works much better when:

- **periods are *integers* (DNA, proteins, …)**
- **datalength is *short***
- ***multiple hidden* periods should be found**

On the lighter side …

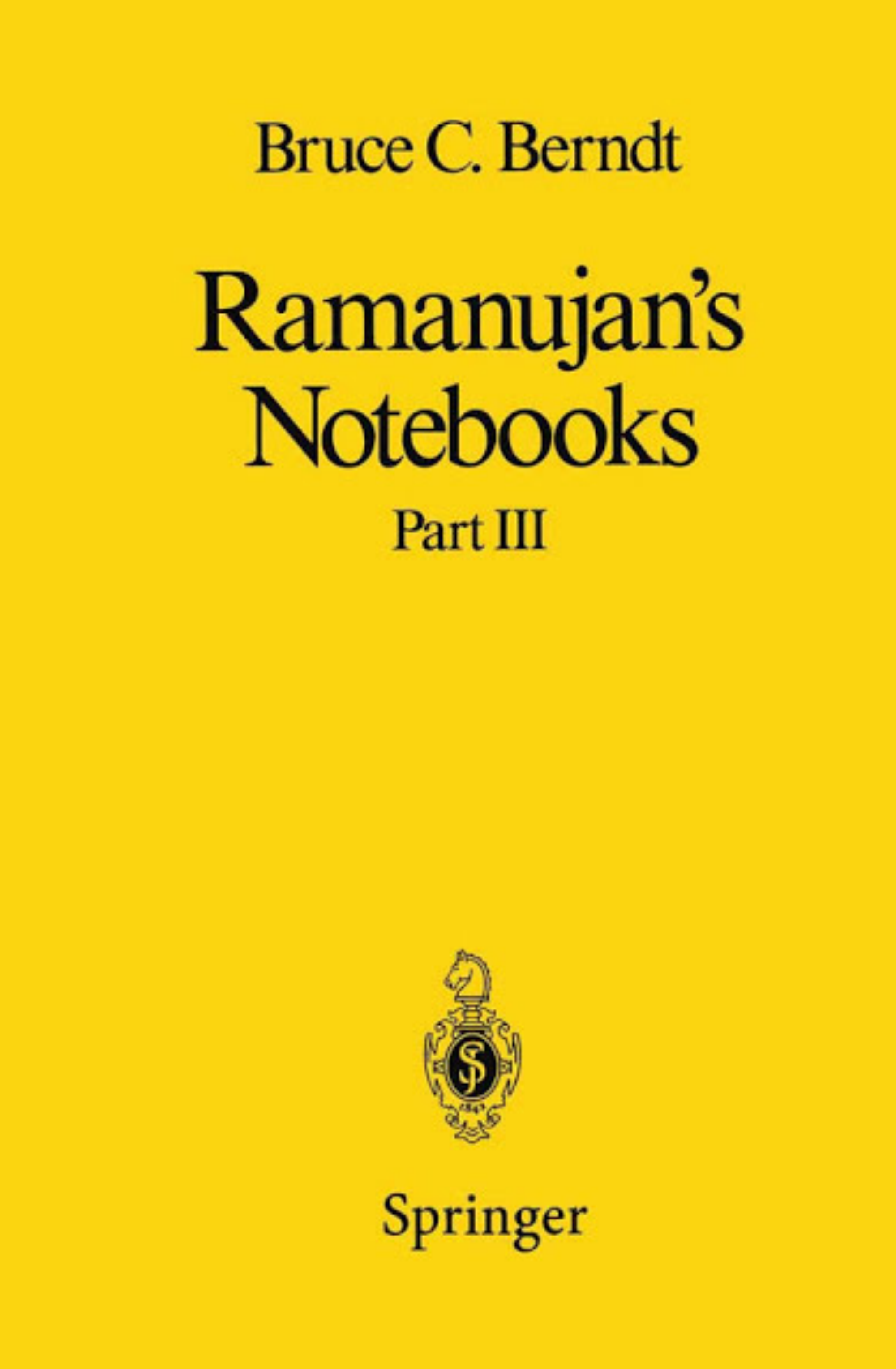
The Taxicab number

$$1729 = 1^3 + 12^3$$

$$= 9^3 + 10^3$$

Smallest integer that can be written as a sum of two cubes in two ways!


CE 1729
Unic

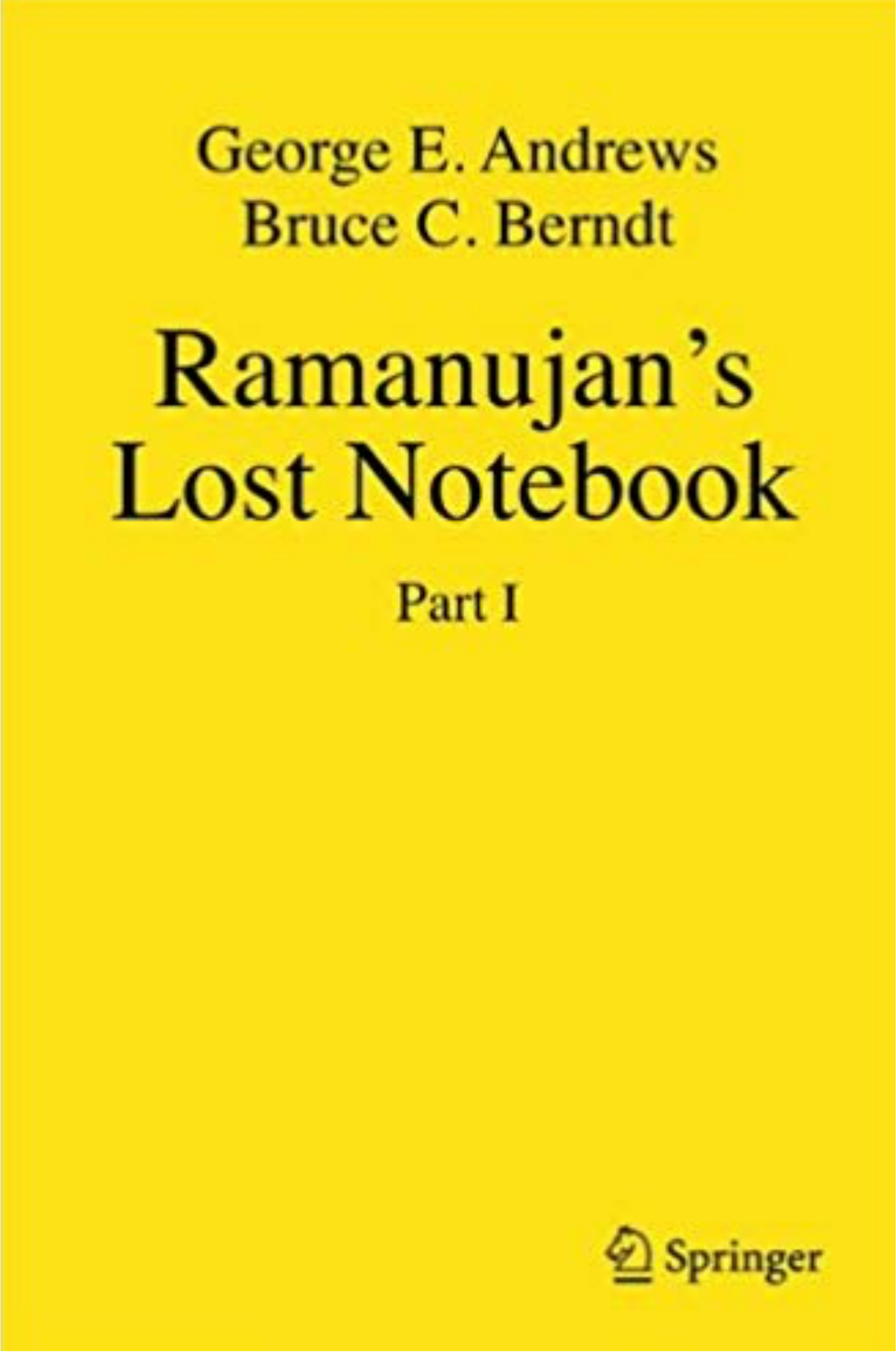


Bruce C. Berndt

Ramanujan's Notebooks

Part III

Springer



George E. Andrews
Bruce C. Berndt

Ramanujan's Lost Notebook

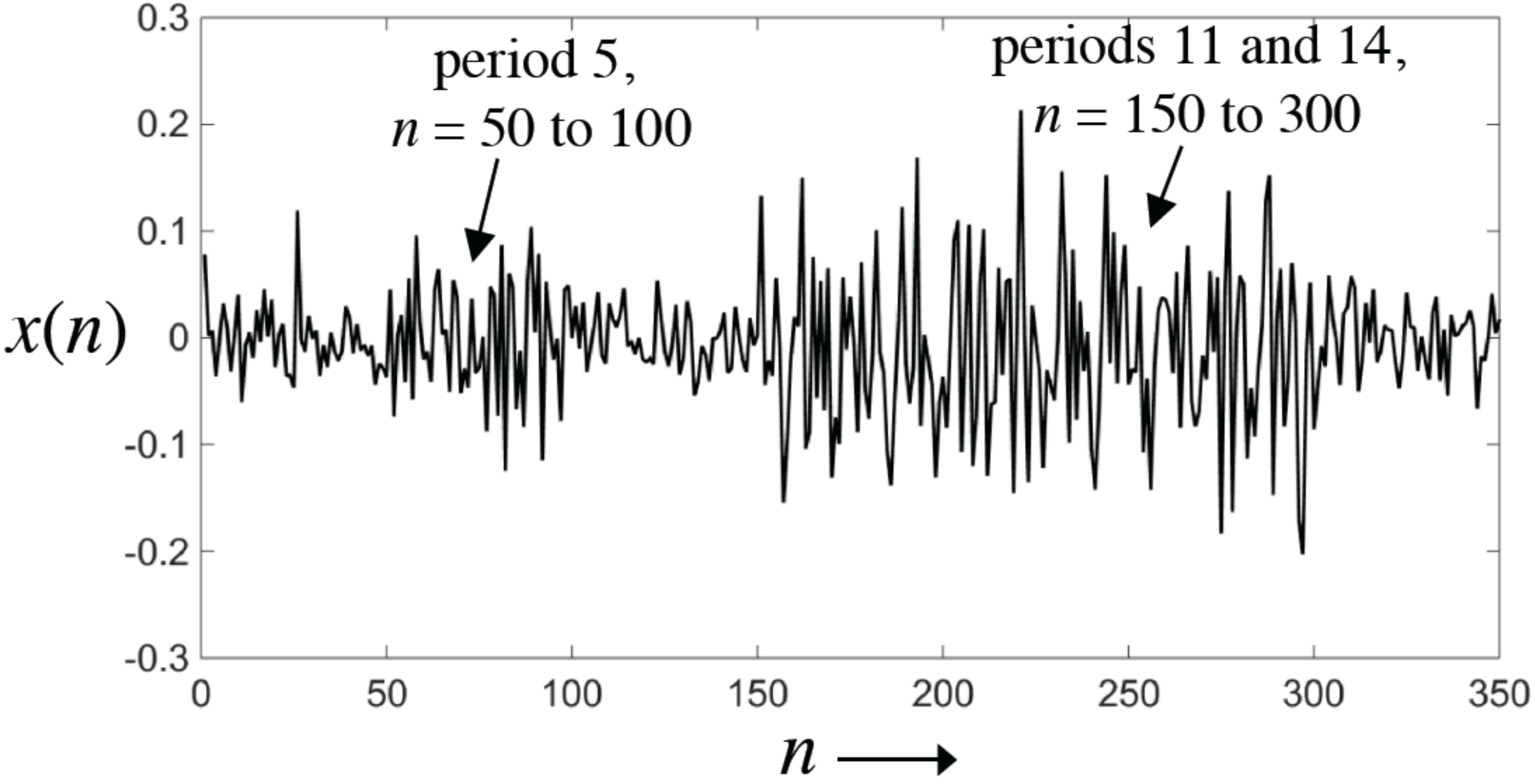
Part I

Springer

Prof. George Andrews
Penn State

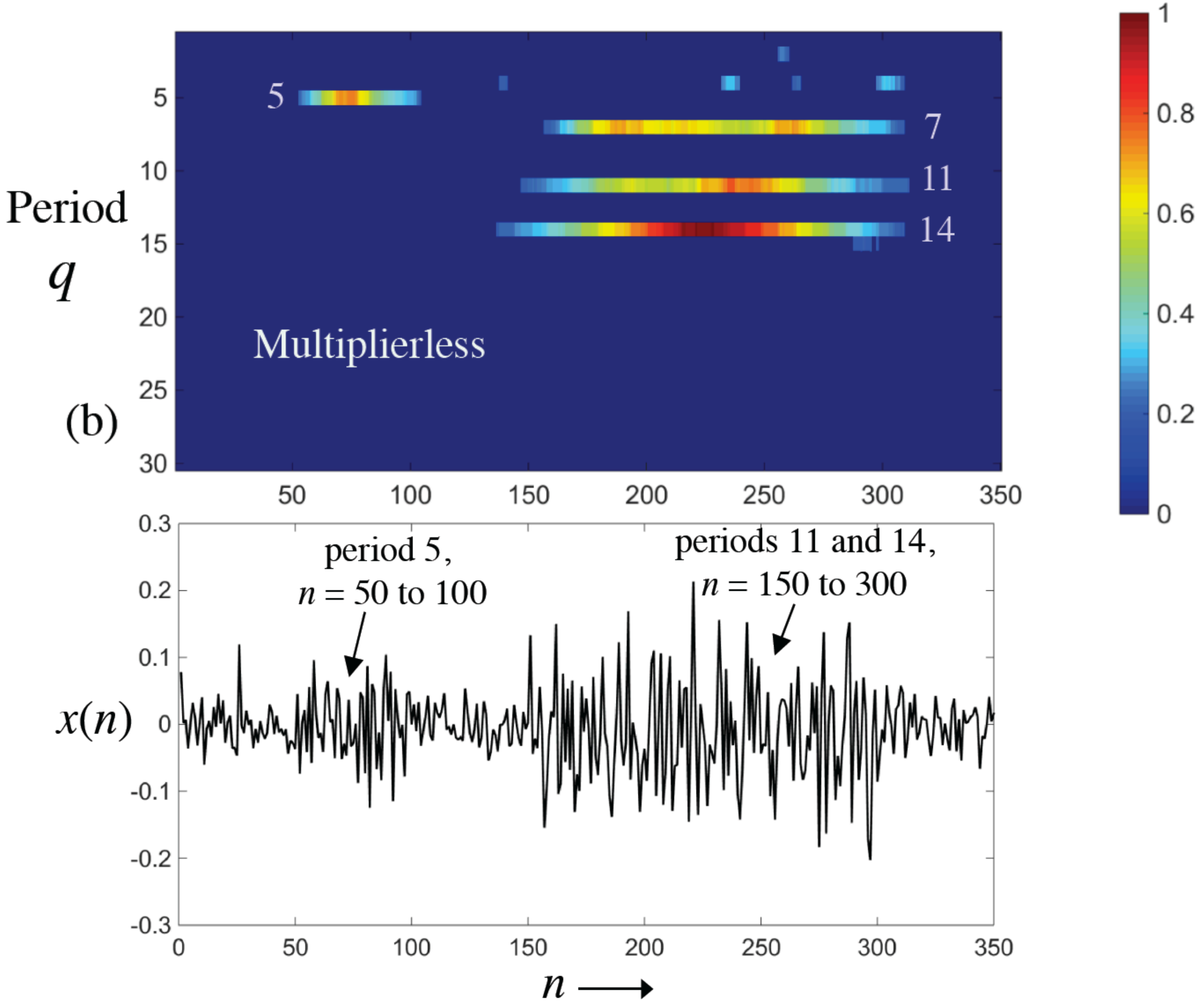
Prof. Bruce Brendt
UIUC

Tracking periodicity as it changes …

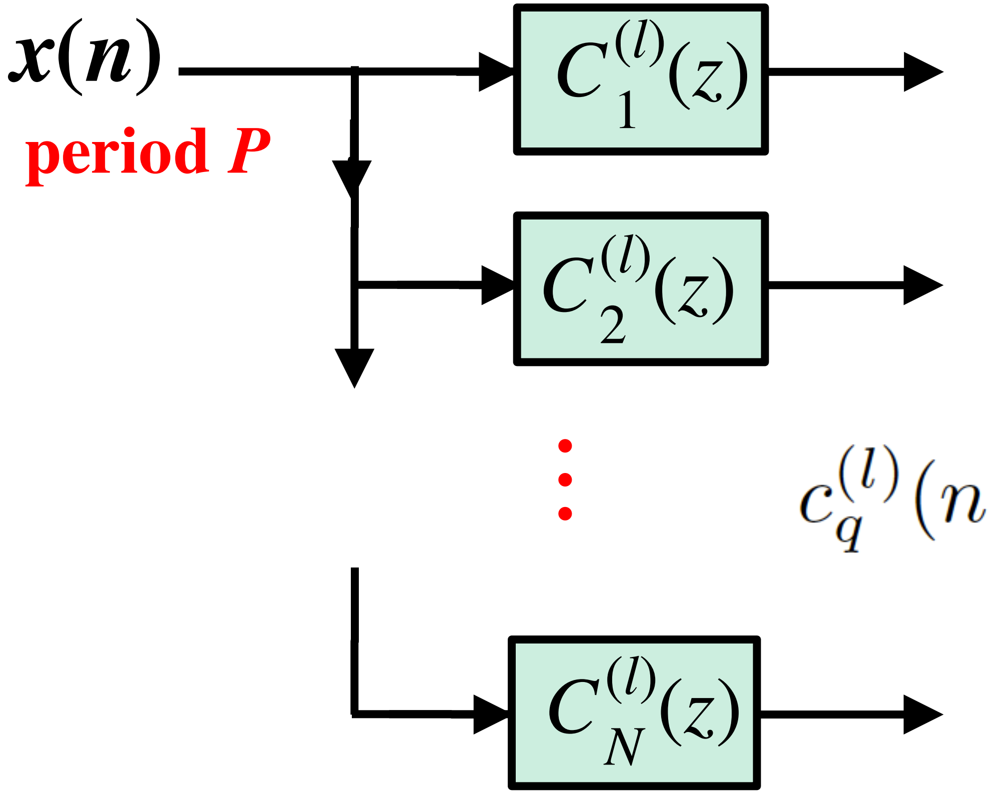


Time-Period plane plot is needed

Ramanujan Filter-Banks



Ramanujan Filter-Banks



$$c_q^{(l)}(n) = \begin{cases} c_q(n) & 0 \le n \le ql - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem: ***Suppose the filters with nonzero outputs are*** $C_{q_1}(z), C_{q_2}(z), \cdots C_{q_K}(z)$

Then $P = \text{lcm}\,\{q_1, q_2, \cdots, q_K\}$

FIR Ramanujan filters

$$c_q^{(l)}(n) = \begin{cases} c_q(n) & 0 \leq n \leq ql - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Can show:

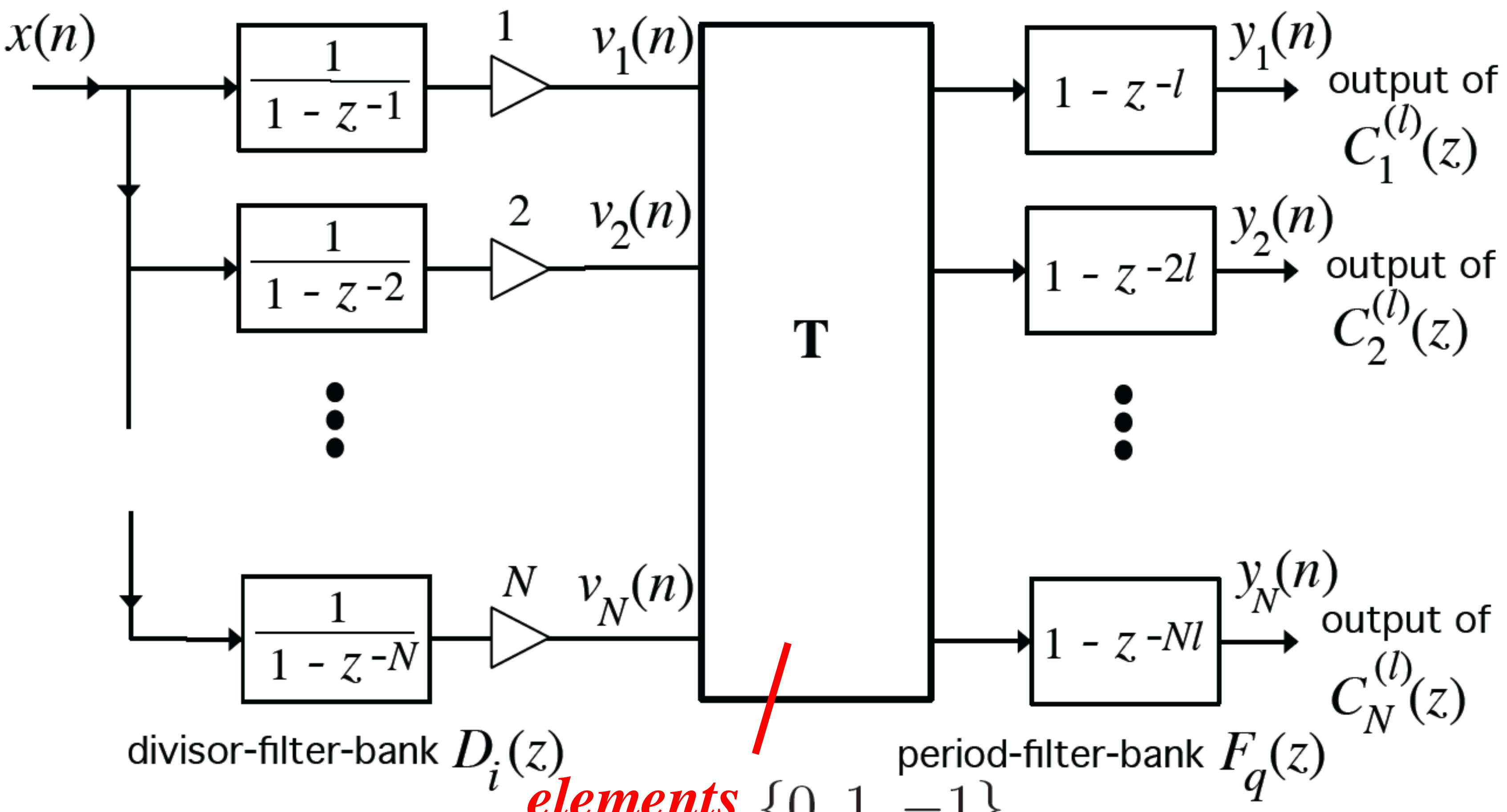
$$C_q^{(l)}(z) = \sum_{q_k | q} \alpha_{q_k} q_k \times \left(\frac{1 - z^{-ql}}{1 - z^{-q_k}} \right)$$

$$\alpha_{q_k} \in \{0, 1, -1\}$$

$d|q$: d is a divisor (or factor) of q

PPV and Tenneti, ICASSP 2017

Multiplierless FIR Ramanujan FB



elements $\{0, 1, -1\}$

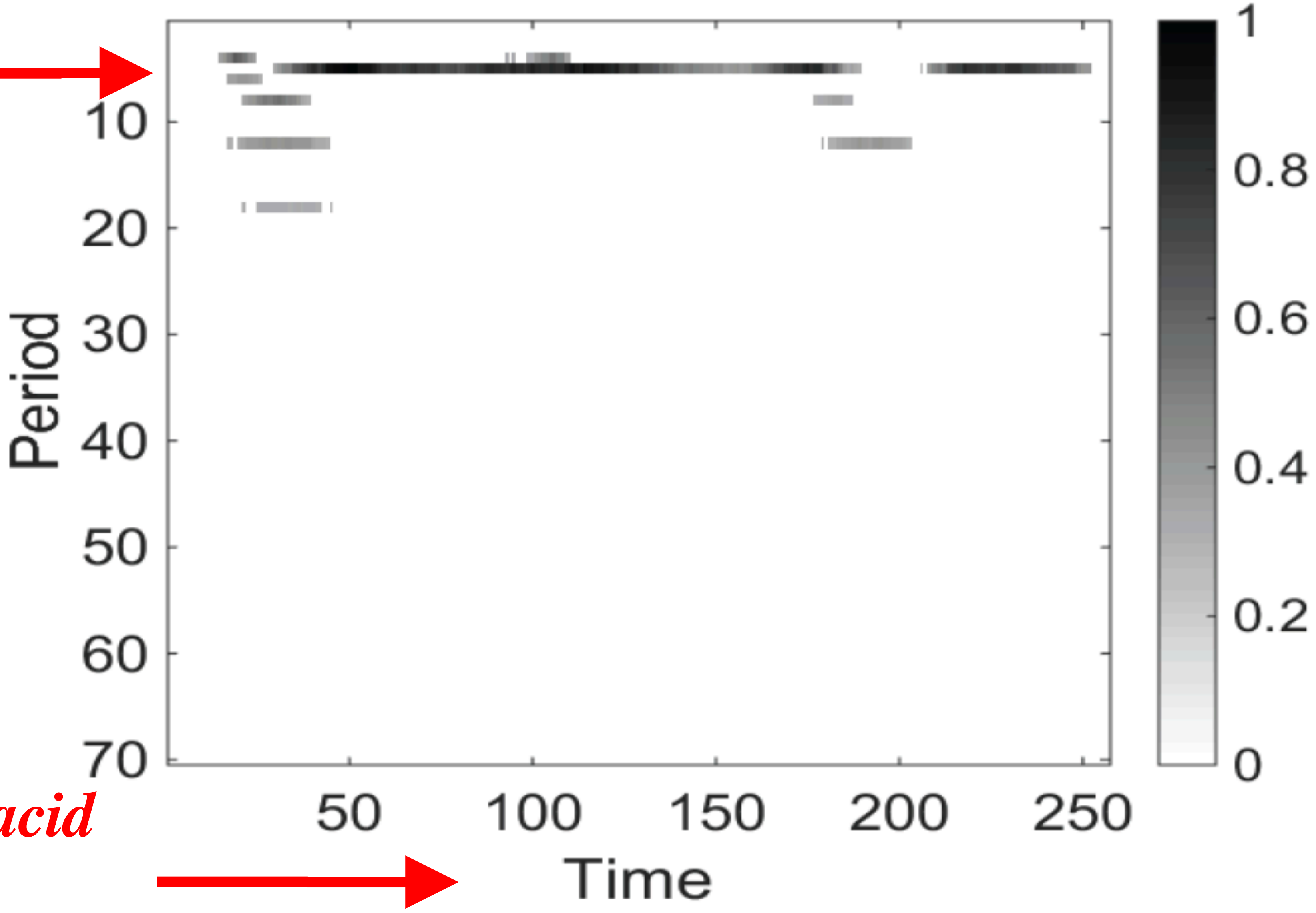
In practice: $D_i(z/\rho) = \left(\frac{1}{1-\rho^i z^{-i}}\right), \quad F_q(z/\rho) = 1 - \rho^{ql} z^{-ql}$

Protein molecules (amino acid sequences)

The HetL protein

- *Has strong period 5 component*
- *Contains insertion loops*
- *Kyte-Doolittle scale, EIIP scale*

Time-period plane from RFB



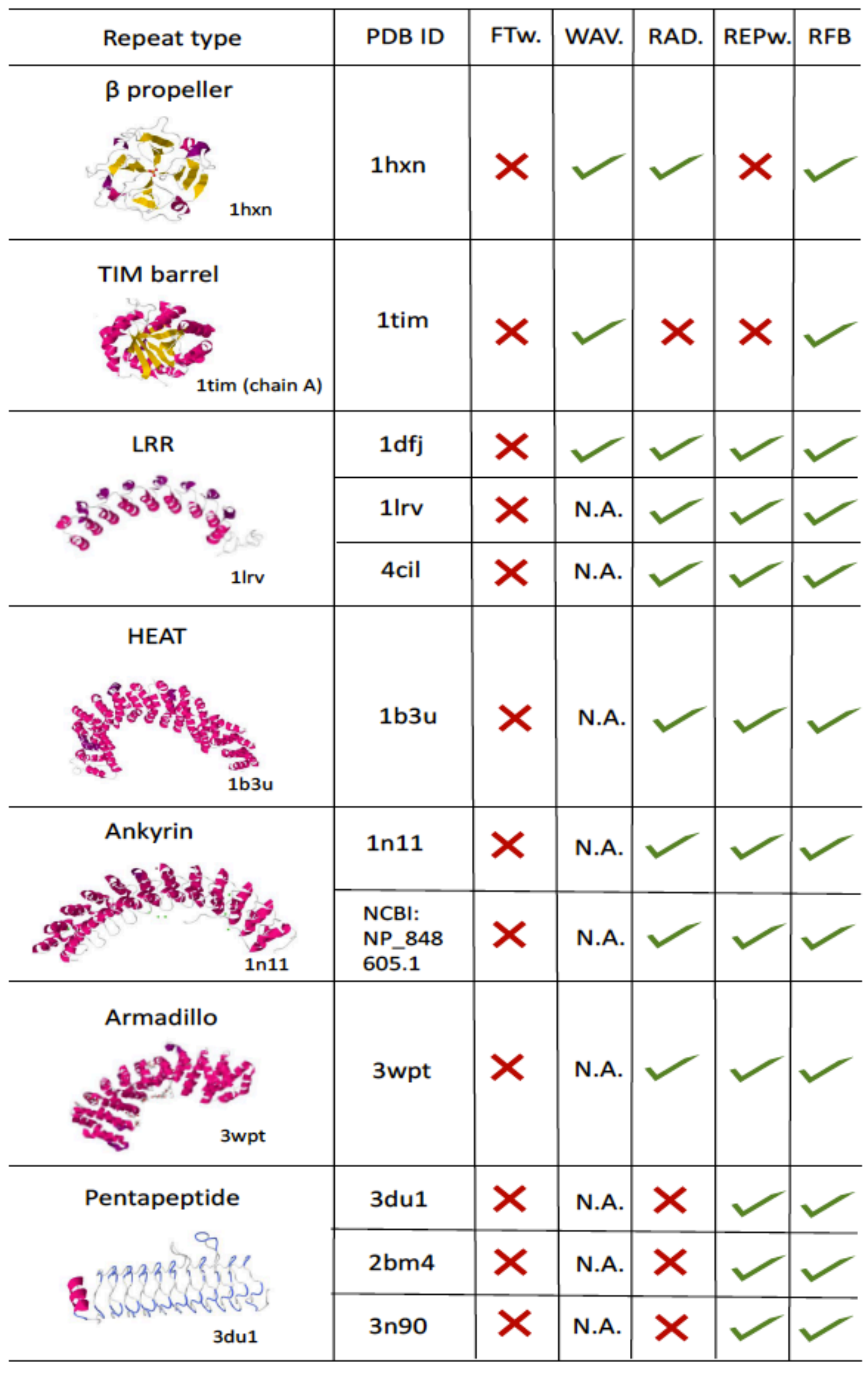
period 5

amino acid domain

More proteins ...

Comparison with other methods …

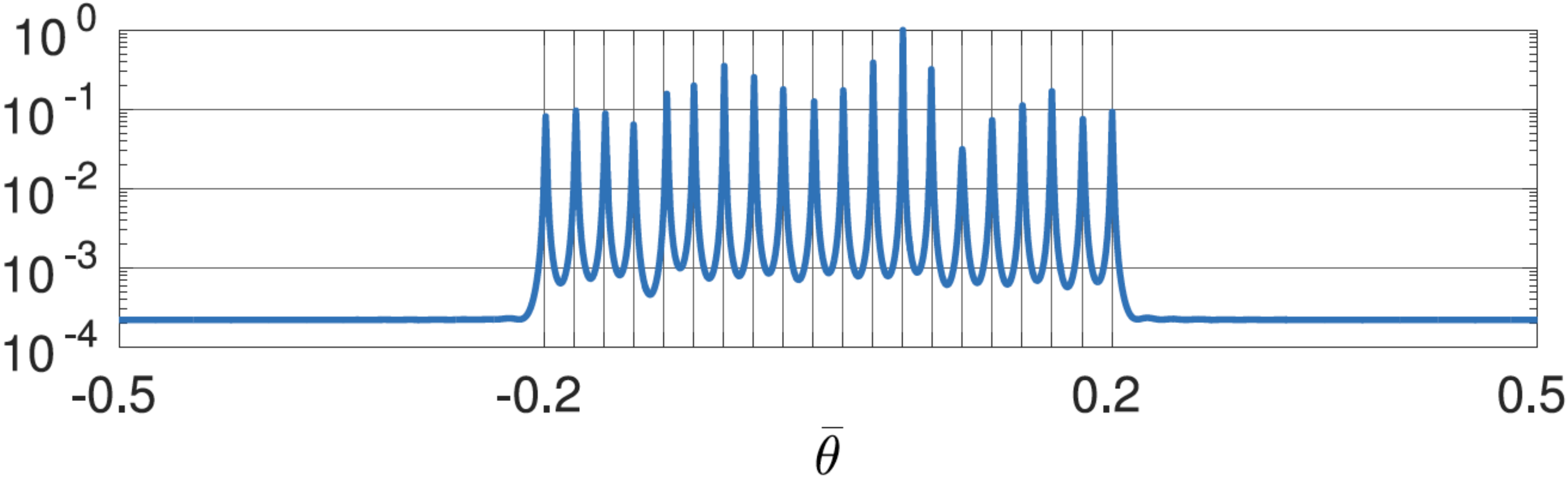
RFB always works



Repeat type	PDB ID	FTw.	WAV.	RAD.	REPw.	RFB
β propeller 1hxn	1hxn	✗	✓	✓	✗	✓
TIM barrel 1tim (chain A)	1tim	✗	✓	✗	✗	✓
LRR 1lrv	1dfj	✗	✓	✓	✓	✓
	1lrv	✗	N.A.	✓	✓	✓
	4cil	✗	N.A.	✓	✓	✓
HEAT 1b3u	1b3u	✗	N.A.	✓	✓	✓
Ankyrin 1n11	1n11	✗	N.A.	✓	✓	✓
	NCBI: NP_848 605.1	✗	N.A.	✓	✓	✓
Armadillo 3wpt	3wpt	✗	N.A.	✓	✓	✓
Pentapeptide 3du1	3du1	✗	N.A.	✗	✓	✓
	2bm4	✗	N.A.	✗	✓	✓
	3n90	✗	N.A.	✗	✓	✓

Tenneti and PPV 2016

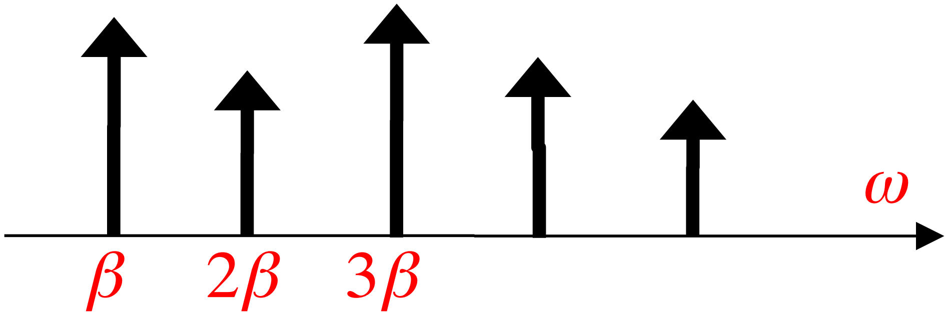
*i*MUSIC



Traditional* MUSIC *spectrum

$x(n)$ periodic: spectrum is harmonic

$$x(n) = \sum_{k=1}^{K} c_k e^{j\omega_k n} + e(n), \;\; 1 \le n \le L$$



Modified MUSIC:

***H*MP,** *Gribonval and Bacry*, 2003.

***H*MUSIC,** *Christensen, Jacobsson and Jensen*, 2006+

More accurate than MUSIC; but complex, time consuming

iMUSIC [Tenneti and PPV, 2017, 2019]

Integer MUSIC (i.e., when period = integer)

$$\mathbf{a}(e^{j\omega}) = \begin{bmatrix} 1 & e^{j\omega} & e^{2j\omega} & \dots \end{bmatrix}^T$$

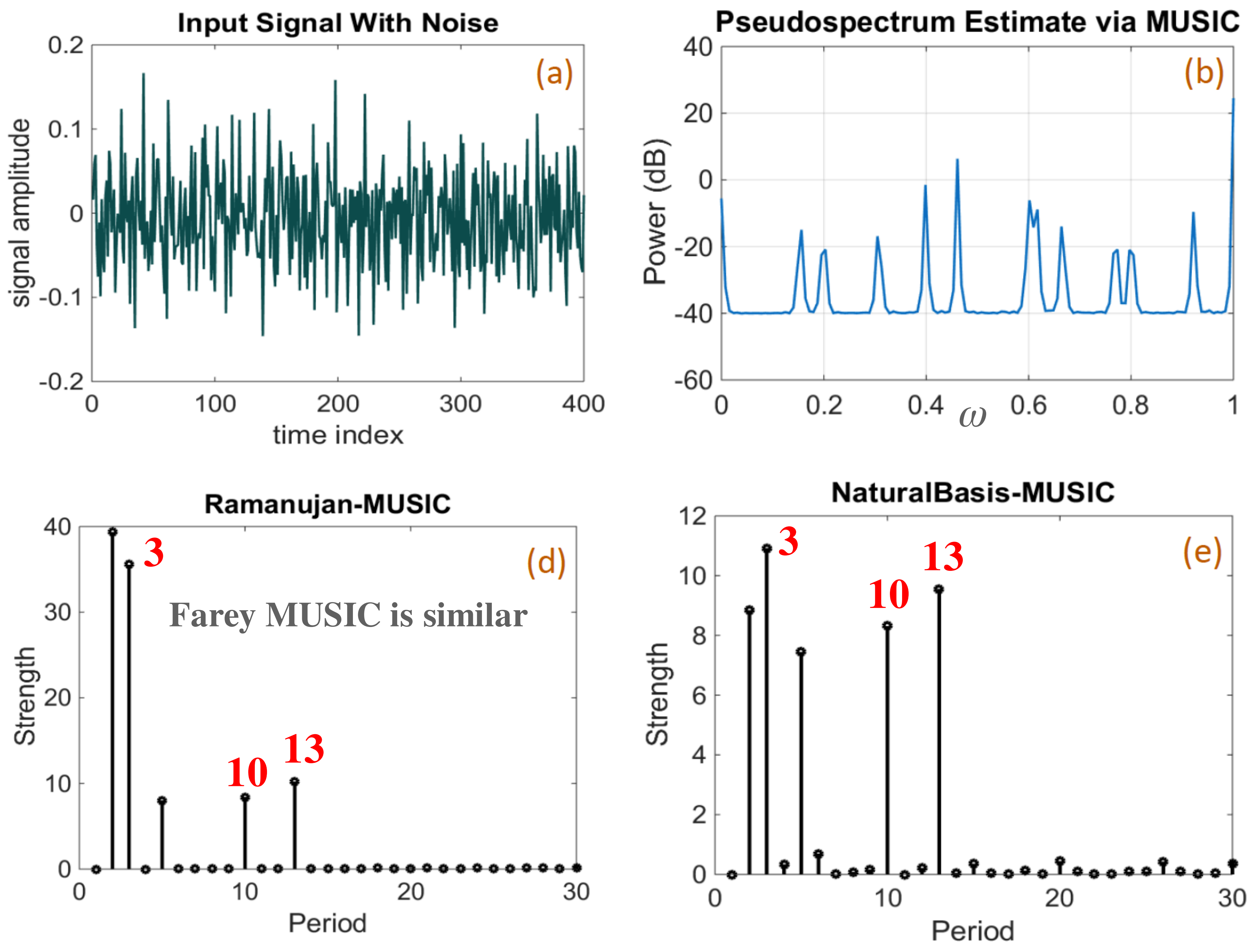
$$\boxed{S_{MU}(e^{j\omega}) = \frac{1}{\mathbf{a}^H(e^{j\omega})\mathbf{U}_e\mathbf{U}_e^H\mathbf{a}(e^{j\omega})}}$$

Instead of this, uses vectors from:

- ***Ramanujan dictionary or***
- ***Farey dictionary or***
- ***Natural basis dictionary***

$$\boxed{\frac{1}{\phi(P)}\sum_{m=1}^{\phi(P)} \frac{1}{\|\mathbf{U_e}^{\dagger}\mathbf{a}_P^{(m)}\|_2^2}}$$

More accurate, much faster …


Input Signal With Noise

Pseudospectrum Estimate via MUSIC

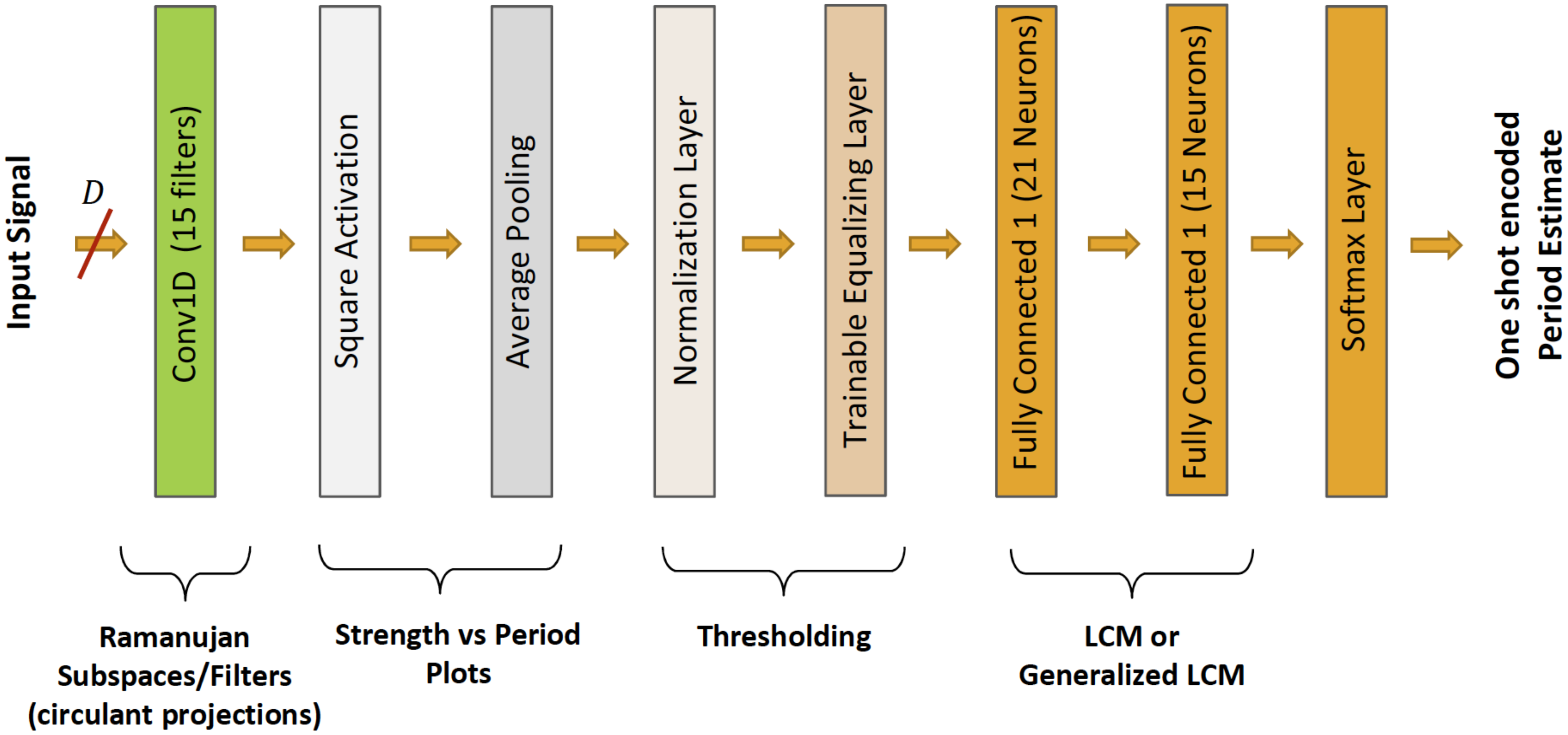
Ramanujan-MUSIC

NaturalBasis-MUSIC

Farey MUSIC is similar

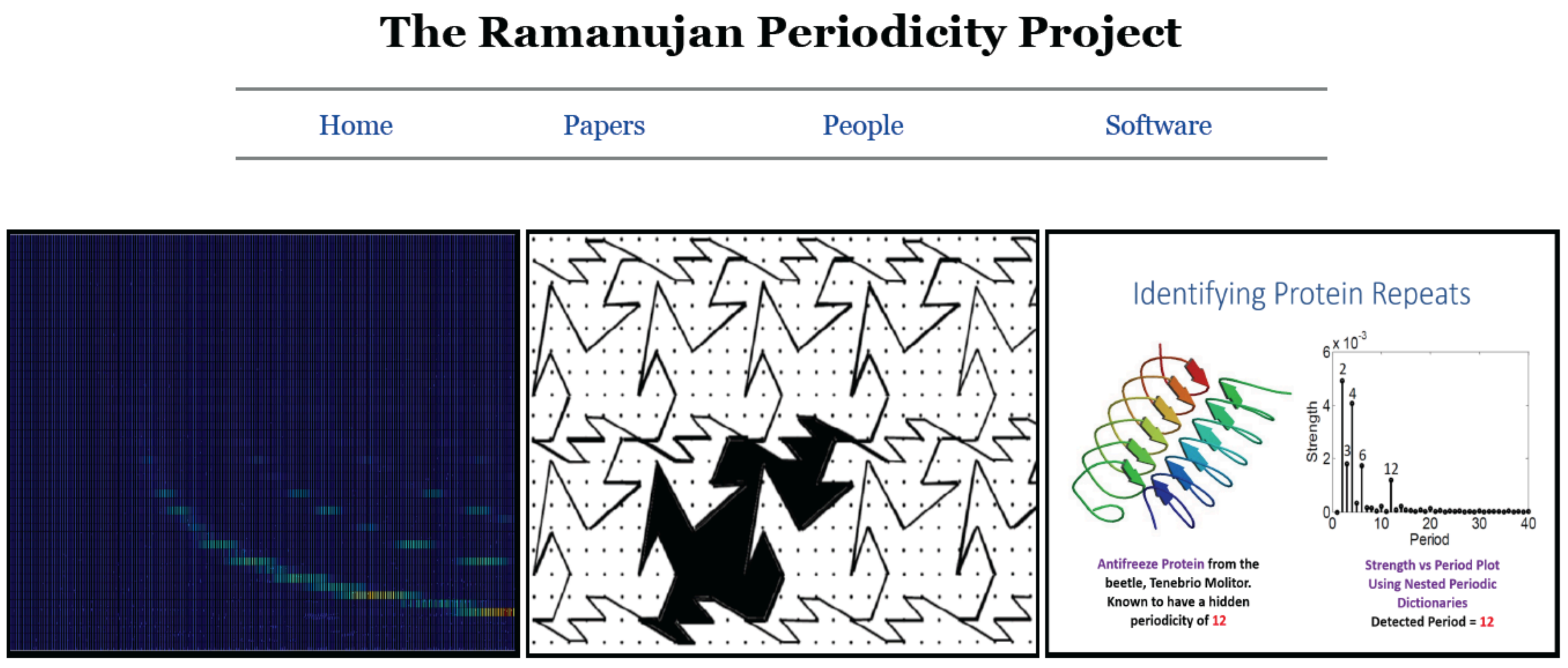
Ongoing and future ….

- Denoising periodic signals
- Non-integer periods
- CNN and Ramanujan
- 2D case


Input Signal
D
Conv1D (15 filters)
Square Activation
Average Pooling
Normalization Layer
Trainable Equalizing Layer
Fully Connected 1 (21 Neurons)
Fully Connected 1 (15 Neurons)
Softmax Layer
One shot encoded Period Estimate
Ramanujan Subspaces/Filters (circulant projections)
Strength vs Period Plots
Thresholding
LCM or Generalized LCM

Our Website on this …

http://systems.caltech.edu/dsp/students/srikanth/Ramanujan/



The Ramanujan Periodicity Project

Home Papers People Software

Period vs Time Plane for a chirp signal using Ramanujan Filter Banks. [View Larger]

Going Beyond Parallelograms! Bringing more general lattices into DSP. [View Larger]

Detecting repeats in bio-molecules such as Proteins using Nested Periodic D

What's it all about?

The Ramanujan Periodicity Project is a new framework for periodicity analysis. Starting from a novel uni for periodic signals, we have developed several techniques to estimate and track periodicities in data. Th methods, projection techniques and filter banks.

Srikanth Tenneti

References for this talk

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2) P. P. Vaidyanathan, “Ramanujan sums in the context of signal processing: Part II: FIR representations and applications,” IEEE Trans. on Signal Proc., Aug., 2014.
3) P. P. Vaidyanathan and Piya Pal, “The Farey dictionary for sparse representation of periodic signals,” Proc. ICASSP, May 2014.
4) S. Tenneti and P. P. Vaidyanathan, “Ramanujan filter banks for estimation and tracking of periodicity properties,” Proc. IEEE ICASSP, April 2015.
5) S. Tenneti and P. P. Vaidyanathan, “Nested Periodic Matrices and Dictionaries: New Signal Representations for Period Estimation,” IEEE Trans. on Signal Proc., July 2015.

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6) P. P. Vaidyanathan and S. Tenneti, “Properties of Ramanujan filter banks,” Proc. EUSIPCO, Sept. 2015.
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13) S. Tenneti and P. P. Vaidyanathan, “DSP-inspired deep learning: a case study using Ramanujan subspaces,” Proc. Asil. Conf. Sig., Sys., and Comp., Monterey, CA, Nov. 2019.
14) P. P. Vaidyanathan and S. Tenneti, “Srinivasa Ramanujan and Signal-Processing Problems,” Philosophical Transactions of the Royal Society, A, Volume 378, Issue 2163, 9 December 2019.
15) P. Kulkarni and P. P. Vaidyanathan, “On the Zeros of Ramanujan Filters,” IEEE Signal Processing Letters, April, 2020.

From *A mathematician’s apology,* 1940

G. H. Hardy
1877 - 1947

The ‘real’ mathematics of the ‘real’ mathematicians is almost wholly ‘useless’.

(So) the ‘real mathematician’ has a clear conscience.

Applied mathematics is ‘useful’, yes. ***But it is trivial.***

Perhaps, Hardy was wrong? ☺

Thank you!