Lecture 10

C Self-Bounding Function (contd. from Lecture 9)

<u>Definition.</u> A function $f: \mathcal{X}^n \to [0, \infty)$ has the self-bounding property if there exist $f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ s.t.

(a) $0 \le f(x) - f_i(x^{-i}) \le 1$;

(b) $\sum_{i=1}^{n} (f(x) - f_i(x^{-i})) \le f(x)$.

Clearly, for self-bounding functions

$$\sum_{i=1}^{n} \left(f(x) - f_i(x^{-i})\right)^2 \le f(x).$$

<u>Corollary.</u> For a self-bounding function f, $Z = f(X)$ satisfies

$$\mathrm{Var}(Z) \le \mathbb{E}[Z].$$

<u>Definition</u> (hereditary property, configuration functions)

A property Π is said to hereditary if when a sequence satisfies it, its subsequence satisfies it as well.

A configuration function is given by $f(x_1, \dots, x_n)$ = length of the longest subsequence satisfying a hereditary property Π.

<u>Lemma</u> A configuration function is self-bounding.

<u>Proof.</u> Let $f_i(x^{-i}) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Then,

$$0 \le f(x) - f_i(x^{-i}) \le 1.$$

Note that for a seq. $x = (x_1, \dots, x_n)$ where the longest subseq. satisfying Π is $(x_{i_1}, \dots, x_{i_k})$, for every $i \notin \{i_1, \dots, i_k\}$ $f(x) - f_i(x) = 0$. Therefore,

$$\sum_{i=1}^{n} f(x) - f_i(x) \le k = f(x). \qquad \blacksquare$$

Examples of self-bounding functions

* longest increasing subsequence
* no. of distinct elements

Example (Maximum eigenvalue)

Consider a symmetric random matrix A with entries X_{ij}, $1 \le i \le j \le n$, that are iid with $|X_{ij}| \le 1$. Let Z denote the max. eigenvalue of A, i.e.,

$$Z = \sup_{u : \|u\| = 1} u^T A u.$$

Then,

$$\begin{aligned}(Z - Z'_{ij})_+ &\le \left(v^T A v - v^T A'_{ij} v\right) \mathbb{1}_{\{Z > Z'_{ij}\}} \\ &= \left(v^T (A - A'_{ij}) v\right) \mathbb{1}_{\{Z > Z'_{ij}\}} \\ &\le 2\left(v_i v_j (X_{ij} - X'_{ij})\right)_+ \\ &\le 4 |v_i v_j|,\end{aligned}$$

where v achieves Z for A. Thus,

$$\sum_{1 \le i \le j \le n} (Z - Z'_{ij})_+^2 \le 16 \sum_{1 \le i \le j \le n} |v_i v_j|^2 \le 16.$$

The Entropy Method

Recall the recipe we used to show the concentration of Z around its mean using the Efron-Stein inequality:

Showed

$$\mathrm{Var}\left(e^{\lambda Z/2}\right) \leq \frac{\lambda^2}{4}\,\mathbb{E}\left[e^{\lambda Z}\sum_{i=1}^{n}(Z-Z_i')_+^2\right]$$

which implied

$$\psi_{Z-\mathbb{E}Z}\left(1/\sqrt{v}\right) \leq \log\frac{16}{9}.$$

Entropy method generalizes this recipe (we shall see in what sense) and yields stronger results than what we obtained earlier.

[A] Step 1: Entropy and Herbst's argument

$$\mathrm{Var}(X) = \mathbb{E}[g(X)] - g(\mathbb{E}[X])$$

for $g(x) = x^2$.

Consider an alternative concave function $h(x) = x\log x$. Let $X \geq 0$ a.s.

$$\boxed{\mathrm{Ent}(X) \overset{\text{def}}{=} \mathbb{E}[h(X)] - h(\mathbb{E}[X])}$$

Lemma (Herbst's argument)

Let Z be a rv with $\mathbb{E}[Z] < \infty$ s.t. there exists $v > 0$ for which

$$(\#) \qquad \frac{\mathrm{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} \le \frac{\lambda^2 v}{2}, \qquad \forall\, \lambda > 0.$$

Then, $\forall\, \lambda > 0$

$$\psi_{Z-\mathbb{E}[Z]}(\lambda) \le \frac{\lambda^2 v}{2}.$$

Proof. The key observation is the following:

$$\psi'(\lambda) = \frac{\mathbb{E}[Ze^{\lambda Z}]}{\mathbb{E}[e^{\lambda Z}]},$$

which implies

$$\frac{\mathrm{Ent}(e^{\lambda Z})}{\mathbb{E}[e^{\lambda Z}]} = \lambda\psi'(\lambda) - \psi(\lambda).$$

Therefore, the Herbst's condition (#) is the same as

$$\lambda\psi'(\lambda) - \psi(\lambda) \le \frac{\lambda^2 v}{2},$$

i.e., for $G(\lambda) = \frac{\psi(\lambda)}{\lambda}$,

$$G'(\lambda) \le \frac{v}{2} \Rightarrow G(\lambda) - G(0) \le \frac{\lambda v}{2}$$

$$G(0) = \lim_{\lambda \to 0} \frac{\psi(\lambda)}{\lambda} = \psi'(0) = 0.$$

Thus, $\psi(\lambda) \le \frac{\lambda^2 v}{2}$. ∎

B Step 2: Tensorization of Herbst's argument

Suppose that we can establish (#) for $n=1$. Then,

$$\frac{\mathrm{Ent}^{(i)}(e^{\lambda Z})}{\mathbb{E}^{(i)}[e^{\lambda Z}]} \le \frac{\lambda^2 v_i}{2}. \quad \left(\mathrm{Ent}^{(i)}(Y) \stackrel{\text{def}}{=} \mathbb{E}^{(i)}[h(Y)] - h(\mathbb{E}^{(i)}[Y])\right)$$

Suppose the counterpart of ES for $\mathrm{Var}(\cdot)$ holds, i.e.,

$$\boxed{\mathrm{Ent}(e^{\lambda Z}) \le \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Ent}^{(i)}(e^{\lambda Z})\right].}$$

Then, the bound for $n=1$ yie

$$\mathrm{Ent}(e^{\lambda Z}) \le \sum_{i=1}^{n} \mathbb{E}\left[\frac{\lambda^2}{2} v_i \, \mathbb{E}^{(i)}[e^{\lambda Z}]\right]$$

$$= \frac{\lambda^2}{2}\left(\sum_{i=1}^{n} v_i\right).$$