B Lower tail bound for weakly self-bounding functions

Theorem $f: \mathcal{X}^n \to [0,\infty)$

- $f(X_1,\dots,X_n)$ be weakly (a,b)-self bounding
- the functions f_i in the self-bounding property satisfy

$$f(x)-f_i(x^{(i)}) \le 1, \quad 1 \le i \le n, \quad \forall\, x \in \mathcal{X}^n$$

Then, $\forall\ 0 < t \le \mathbb{E}Z$

$$\boxed{\mathbb{P}(Z \le \mathbb{E}Z - t) \le \exp\left(\frac{-t^2}{2(a\mathbb{E}Z + b + c_- t)}\right)}$$

where $c = (3a-1)/6$ and $c_- = \max\{0, -c\}$.

Proof. We restrict to the case $c_- = 0$ (namely for the case $a \ge 1/3$); the general case is essentially the same but is more ugly.

For the lower tail, consider $Y = -Z$, $Y_i = -Z_i$. Then, the MLSI implies that for $\lambda > 0$

$$\begin{aligned}
\mathrm{Ent}(e^{\lambda Y}) &\le \sum_{i=1}^n \mathbb{E}\left[e^{\lambda Y}\, \phi(-\lambda(Y-Y_i))\right] \\
&\le \sum_{i=1}^n \mathbb{E}\left[e^{\lambda Y}\phi(\lambda)(Y_i - Y)^2\right]
\end{aligned}$$

(since $(Y_i - Y) \le 1$ and $\lambda > 0$; we use $\phi(\lambda x) \le \phi(\lambda) x^2$, $x \in (0,1)$)

$$\le \phi(\lambda)\, \mathbb{E}\left[e^{\lambda Y}(aZ + b)\right]$$

Substituting $G(\lambda) = \psi_{Y - \mathbb{E}Y}(\lambda)$

$$\boxed{(\lambda + a\,\phi(\lambda))\,G'(\lambda) - G(\lambda) \le v\,\phi(\lambda),} \quad (\#)$$

where $v = a\,\mathbb{E}Z + b$.

→ Solving (#) is not easy. We shall show:

$$G(\lambda) \le v\frac{\lambda^2}{2}, \qquad \forall\, \lambda \in \left[0, \frac{1}{a}\right)$$

Thus,

$$\mathbb{P}(Y - \mathbb{E}Y > t) \le e^{-\lambda t + G(\lambda)}$$
$$\le e^{-\lambda t + \frac{v\lambda^2}{2}} \qquad \forall\, \lambda \in \left[0, \frac{1}{a}\right)$$

If $t \le \frac{v}{a}$, we can choose $\lambda = \frac{t}{v}$ to get

$$\mathbb{P}(Y - \mathbb{E}Y > t) \le e^{-t^2/2v}.$$

It remains to prove the following:

Lemma. For $a \ge 1/3$, $v > 0$, suppose that G satisfies the differential inequality

$$(\lambda + a\phi(\lambda))\,H'(\lambda) - H(\lambda) \le v\,\phi(\lambda).$$

Then, $\forall\, \lambda \in [0, 1/a)$, $G(\lambda) \le v\lambda^2/2$.

Background:

Gronwall's inequality

Continuous u, v s.t.

$$u(t) \le c + K\int_0^t u(s)\,v(s)\,ds \qquad \forall\, t \in [0, T].$$

Then, $u(t) \le c\,\exp\left(K\int_0^t v(s)\,ds\right)$.

$$\Big(\dot{u}(t) \le K u(t) v(t) \Leftrightarrow \frac{d}{dt} \ln u(t) \le K v(t)$$

$$\Rightarrow \ln(u(t)/u(0)) \le K \int_0^t v(s)\, ds$$

$$\Rightarrow u(t) \le u(0) \exp\left(K \int_0^t v(s) ds\right).$$

Since

$$u(0) \le C, \quad u(t) \le C \exp\left(K \int_0^t v(s) ds\right). \Big)$$

Proof. $f(t) \overset{\text{def}}{=} \int_0^t u(s) v(s)\, ds$

$$\dot{f}(t) = u(t) v(t) \le C v(t) + K v(t) f(t)$$

$$g(t) = \exp\left(-K \int_0^t v(s)\, ds\right) f(t)$$

Then, $\dot{g}(t) \le \frac{d}{dt}\left(-\frac{C}{K} e^{-K \int_0^t v(s) ds}\right)$

$$\Rightarrow g(t) - g(0) \le \frac{C}{K}\left[1 - e^{-K \int_0^t v(s) ds}\right]$$

$$\Rightarrow f(t) \le \frac{C}{K}\left[\exp\left(K \int_0^t v(s) ds\right) - 1\right]$$

Thus,

$$u(t) \le C + K f(t)$$
$$\le C \exp\left(K \int_0^t v(s)\, ds\right).$$

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We need a similar result:

Lemma A. $f(t) \dot{G}(t) - \dot{f}(t) G(t) \le f^2(t) g(t), \ \forall\, t \in I$

Assume: $f(0) = 0$, $\dot{f}(t) > 0$, $f(x) \ne 0 \ \forall\, x \ne 0$

g continuous on I

$G(0) = \dot{G}(0) = 0$; G twice differentiable

Then, $\forall\, t \in I$, $G(t) \le f(t) \int_0^t g(s)\, ds$.

<u>Proof.</u> Let $\rho(t) = G(t)/f(t)$ $\forall\, t \neq 0$ and $\rho(0) = 0$.

Then, $\dot{\rho}(t) = \dfrac{f(t)\dot{G}(t) - \dot{f}(t)G(t)}{f^2(t)}$ $\forall\, t \neq 0$

and

$$\dot{\rho}(0) = \lim_{t\to 0} \frac{\dot{f}(t)\dot{G}(t) + f(t)\ddot{G}(t) - \ddot{f}(t)G(t) - \dot{f}(t)\dot{G}(t)}{2 f(t)\dot{f}(t)}$$

$$= \frac{\ddot{G}(0)}{2\dot{f}(0)}. \quad (\rho \text{ is continuously differentiable}).$$

By our condition, $\dot{\rho}(t) \leq g(t)$ $\forall\, t \in I,\ t \neq 0$.

$$\Rightarrow \rho(t) \leq \int_0^t g(s)\,ds \quad \forall\, t \neq 0$$

$$\Rightarrow G(t) \leq f(t) \int_0^t g(s)\,ds \quad \forall\, t \neq 0. \qquad \blacksquare$$

<u>Lemma 8.</u> (Key Lemma)

Let ρ be continuous on $[0, \theta)$. Suppose that for $a \geq 0$ and $v > 0$, a twice differentiable function H satisfies

$$\lambda H'(\lambda) - H(\lambda) \leq \rho(\lambda)\left(-a H'(\lambda) + v\right)$$

s.t. $-a H'(\lambda) + v > 0$ $\forall\, \lambda \in [0, \theta)$

$$H'(0) = H(0) = 0$$

Let $\rho_0 : [0, \theta) \to \mathbb{R}$; $G_0 : [0, \theta) \to \mathbb{R}$ twice diff. s.t. $\forall \lambda \in [0,\theta)$, $-a G_0'(\lambda) + 1 > 0$, $G_0'(0) = G_0(0) = 0$, $G_0''(0) = 1$

Suppose that $\lambda G_0'(\lambda) - G_0(\lambda) = \rho_0(\lambda)\left(-a G_0'(\lambda) + 1\right)$.

If $\rho(\lambda) \leq \rho_0(\lambda)$ $\forall\, \lambda \in [0, \theta)$, then $\boxed{H \leq v G_0}$

Proof.

$$\lambda H'(\lambda) - H(\lambda) \le \frac{(\lambda G_0'(\lambda) - G_0(\lambda))}{(-a G_0'(\lambda) + 1)} \cdot (-a H'(\lambda) + v)$$

$$\Leftrightarrow \left[\lambda + \frac{a(\lambda G_0'(\lambda) - G_0(\lambda))}{1 - a G_0'(\lambda)}\right] H'(\lambda) - H(\lambda)$$

$$\le \frac{\lambda G_0'(\lambda) - G_0(\lambda)}{1 - a G_0'(\lambda)} \cdot v$$

$$\Leftrightarrow (\lambda - a G_0(\lambda)) H'(\lambda) - (1 - a G_0'(\lambda)) H(\lambda)$$

$$\le (\lambda G_0'(\lambda) - G_0(\lambda)) v$$

Let $f(\lambda) = \lambda - a G_0(\lambda)$

$$g(\lambda) = v(\lambda G_0'(\lambda) - G_0(\lambda)) / f^2(\lambda)$$

$$\frac{f(\lambda) G_0'(\lambda) - f'(\lambda) G_0(\lambda)}{f(\lambda)^2}$$

$$\ni f(\lambda) H'(\lambda) - f'(\lambda) H(\lambda) \le g(\lambda) f^2(\lambda)$$

By Lemma A,

$$H(\lambda) \le f(\lambda) \int_0^\lambda g(x)\, dx$$

$$= v \cdot f(\lambda) \cdot \int_0^\lambda \frac{G_0'(x) - G_0(x)}{(\lambda - a G_0(x))^2}\, dx$$

$$= v \cdot f(\lambda) \cdot \int_0^\lambda \left(\frac{G_0(x)}{f(x)}\right)' dx$$

$$= v \cdot G(\lambda) \quad \left(\text{since } \lim_{x \to 0} \frac{G_0(x)}{f(x)} = 0\right).$$

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