Lecture 20

$$\alpha(t) = \sup_{A \text{ s.t. } P(A) \geq 1/2} P(A_t^c)$$

Levy's inequalities: For a Lipchitz function f and $Z = f(X)$,

$$\mathbb{P}(Z \geq \mathbb{M}Z + t) \leq \alpha(t)$$

$$\mathbb{P}(Z \leq \mathbb{M}Z - t) \leq \alpha(t)$$

We saw that exact isoperimetry results yield bounds for $\alpha(t)$. (e.g., Uniform distribution over S^{n-1}: $\alpha(t) \leq e^{-(n-1)t^2/2}$
Gaussian over $\mathbb{R}^n$: $\alpha(t) \leq e^{-t^2/2}$)

→ We shall see that often it is easier to get bounds on $\alpha(t)$ without an exact isoperimetry result.

$\boxed{C}$ Bounded difference property and isoperimetry

Consider $f : \mathcal{X}^n \to \mathbb{R}$ s.t.

$$f(x) - f(y) \leq \sum_{i=1}^{n} c_i \mathbb{1}_{\{x_i \neq y_i\}} =: d_c(x, y)$$

Thus, f is Lipchitz under $d_c(x,y)$.

For this metric, $$\alpha(t) = \sup_{A : P(A) \geq 1/2} P(A_t^c) = \sup_{A : P(A) \geq 1/2} P(d_c(X, A) > t)$$

Theorem. For every $c = (c_1, \dots, c_n)$ and $A \subseteq \mathcal{X}^n$

$$P(A) . P(d_c(X, A) \geq t) \leq e^{-t^2/2\|c\|_2^2}.$$

Corollary 1. For $d_c(x,y)$, $\alpha(t) \leq 2 \cdot e^{-t^2/2\|c\|_2^2}$.

Thus, for a function f satisfying BDP with $\underline{c}$,

$$\mathbb{P}(Z > \mathbb{M}Z + t) \leq 2e^{-t^2/2\|c\|_2^2}$$

$$\mathbb{P}(Z < \mathbb{M}Z - t) \leq 2e^{-t^2/2\|c\|_2^2}.$$

Corollary 2. (Blowing-up Lemma)

For Hamming distance $d_H(x,y)$,

$$\mathbb{P}(d_H(X,A) > nt) \leq e^{-n\left(\frac{t^2}{2} - \frac{1}{n}\log\frac{1}{P(A)}\right)}.$$

In particular, for every $\beta_n \to 0$, there exits $t_n, \delta_n \downarrow 0$ s.t.

$$P(X \in A) \geq \exp(-n\beta_n) \; \forall \, n \text{ large}$$

$$\Rightarrow P(d_H(X,A) > n t_n) \leq \delta_n.$$

Proof. Choose $t^2 = 2\left(\beta_n + \log n/n\right)$. Then,

$$\mathbb{P}\left(d_H(X,A) > n\sqrt{2(\beta_n + \log n/n)}\right) \leq \frac{1}{n}. \qquad \blacksquare$$

Proof of Theorem

We shall show:

$$\boxed{\mathbb{P}\left(d_c(X,A) \geq t + \sqrt{\frac{\|c\|_2^2}{2}\log\frac{1}{P(A)}}\right) \leq e^{-2t^2/\|c\|_2^2}} \quad (\#)$$

Then, denoting $u = \sqrt{\frac{\|c\|_2^2}{2}\log\frac{1}{P(A)}}$, for every $t \geq u$,

$$\mathbb{P}\left(d_c(X,A) \geq t\right) \leq e^{-2(t-u)^2/\|c\|_2^2}.$$

Since $(t-u)^2 \geq t^2/4$ for every $t \geq 2u$,

$$\mathbb{P}(d_c(X,A) \geq t) \leq e^{-t^2/2\|c\|_2^2}, \quad \forall\ t \geq 2u.$$

On the other hand, $\quad t < 2u \Leftrightarrow P(A) \leq e^{-t^2/2\|c\|_2^2}.$

Thus, $\forall\, t>0$, $\mathbb{P}(A)\,\mathbb{P}(d_c(X,A) \geq t) \leq e^{-t^2/2\|c\|_2^2}$.

We now complete the proof of $(\#)$. The proof strategy follows Talagrand and uses concentration around mean. Thus, interestingly, conc. around mean yields conc. around median. Indeed, noting that $g(x) = d_c(x,A)$ satisfies BDP with constants c,

$$\mathbb{P}\left(d_c(X,A) \leq \mathbb{E}\, d_c(X,A) - t\right) \leq e^{-2t^2/\|c\|_2^2}.$$

Thus, $\mathbb{P}(d_c(X,A) \leq 0) = \mathbb{P}(A) \leq e^{-2(\mathbb{E}\, d_c(X,A))^2/\|c\|_2^2}$.

Using the other direction,

$$\mathbb{P}\left(d_c(X,A) > \mathbb{E}\, d_c(X,A) + t\right) \leq e^{-2t^2/\|c\|_2^2}$$

we get

$$\mathbb{P}\left(d_c(X,A) > \sqrt{\frac{\|c\|_2^2}{2}\log\frac{1}{P(A)}} + t\right) \leq e^{-2t^2/\|c\|_2^2}. \quad \blacksquare$$

$\boxed{D}$ <u>Going beyond BDP: Talagrand's Convex Distance Ineq.</u>

An examination of the proof of Levy's inequalities reveals that it does not require d to be a metric. We now greatly extend Corollary 1 above to functions which are Lipchitz in a more "relaxed" notion of distance than

$d_c(x,y)$. This distance is known as Talagrand's convex distance and is given by

$$d_T(x,A) = \sup_{c:\|c\|_2=1} \inf_{y\in A} d_c(x,y)$$

To obtain the counterpart of Corollary 1 for $d_T(x,y)$, we need the counterpart of the Theorem above for it. This is precisely the Talagrand's convex distance inequality.

Note that Theorem above says that

$$\mathbb{P}(A) \sup_{c:\|c\|_2=1} \mathbb{P}(d_c(X,A) > t) \le e^{-t^2/2}.$$

The result below says that the supremum can be taken inside $\mathbb{P}$.

Theorem (Talagrand's Convex Distance Inequality)

$$\boxed{\mathbb{P}(A)\, \mathbb{P}(d_T(X,A) \ge t) \le e^{-t^2/4}.}$$

We can now extend Corollary 1 to functions that are Lipchitz in d_T. However, d_T is not a metric: It only defines distance of a point to a set and

$$\sup_{y\in A} d_T(x,y) \ne d_T(x,A).$$

But a simple generalization of the definition of Lipchitz will enable the required extension. Specifically, we say f is Lipchitz in a "set distance" d if $$f(x) \leq \sup_{y \in A} f(y) + d(x, A) \quad \forall A \subseteq \mathcal{X}^n.$$

Then, since for set $A = \{x : f(x) \leq \mathbb{M} f(X)\}$, $A_t \subseteq \{y : f(y) \leq \mathbb{M} f(X) + t\}$, Levy's inequalities extend to this definition of Lipchitz. We can therefore obtain the following extension of Corollary 1.

<u>Corollary 3</u>: Let $f : \mathcal{X}^n \longrightarrow \mathbb{R}$ be Lipchitz in d_T. Then, for $X = X_1, \dots, X_n$ indep.,
$$\mathbb{P}\left(f(X) > \mathbb{M} f(X) + t\right) \leq 2 \cdot e^{-t^2/4}$$
and
$$\mathbb{P}\left(f(X) < \mathbb{M} f(X) - t\right) \leq 2 \cdot e^{-t^2/4}.$$

<u>Lemma</u>. If for all x $\exists\, c$ s.t. $\|c\|_2 \leq 1$, $c_i \geq 0$, s.t. $f(x) \leq f(y) + d_c(x, y)$ $\forall y$, then f is Lipchitz in d_T.