Lecture 21

$$d_T(x,y) = \sup_{c:\|c\|_2 \le 1} d_c(x,y)$$: Talagrand's convex distance

Talagrand's Convex Distance Inequality

$$\mathbb{P}(X^n \in A)\, \mathbb{P}(d_T(X^n, A) > t) \le e^{-t^2/4}$$

Corollary: For a function $f: \mathcal{X}^n \to \mathbb{R}$ which is Lipchitz in d_T, $\mathbb{P}(f(X^n) > \mathbb{M}f(X^n) + t) \le 2 \cdot e^{-t^2/4}$

$\hookrightarrow$ $f(x) \le \sup_{y \in A} f(y) + d_T(x, A)$

$\boxed{E}$ Proof of Talagrand's convex distance inequality

Shall show a weaker version:

$$\boxed{\mathbb{P}(X^n \in A)\, \mathbb{P}(d_T(X^n, A) > t) \le e^{-t^2/10}}$$

Lemma: Denoting $g(x) = d_T(x, A)$.

* $g(x)$ is $(4,0)$-weakly-self bounding with $g(x) - g_i(x^{(i)}) \le 1$.

Then, using the left-tail bound for weakly-self bounding functions

$$\mathbb{P}(d_T(X,A)^2 \le \mathbb{E}[d_T(X,A)^2] - t)$$

$$\le \exp\left(\frac{-t^2}{2 \cdot 4 \cdot \mathbb{E}[d_T(X,A)^2]}\right).$$

In particular,

$$\mathbb{P}(A) = \mathbb{P}(d_T(X,A)^2 \le 0) \le \exp\left(-\frac{\mathbb{E}[d_T(X,A)^2]}{8}\right).$$

Also,

$$\log \mathbb{E} e^{\lambda(d_T(X,A)^2 - \mathbb{E}[d_T(X,A)^2])} \leq \frac{2\mathbb{E}[d_T(X,A)^2]\lambda^2}{(1-2\lambda)}.$$

Choosing $\lambda = \frac{1}{10}$,

$$\mathbb{E}\left[\exp\left(\frac{d_T(X,A)^2}{10}\right)\right] \leq \exp\left(\frac{\mathbb{E}[d_T(X,A)^2]}{8}\right)$$

Thus, $$\mathbb{P}(d_T(X,A) > t) \leq e^{-t^2/10} \cdot \mathbb{E}\left[\exp\left(\frac{d_T(X,A)^2}{10}\right)\right]$$
$$\leq e^{-t^2/10} \cdot \exp\left(\frac{\mathbb{E}[d_T^2(X,A)]}{8}\right)$$

which gives

$$\mathbb{P}(A)\,\mathbb{P}(d_T(X,A) > t) \leq e^{-t^2/10}. \qquad \blacksquare$$

Proof of Lemma

Alternative form of d_T

$$d_T(x,A) = \sup_{c : \|c\|_2 \leq 1} \inf_{y \in A} \sum_{i=1}^n c_i \mathbb{1}_{\{x_i \neq y_i\}}$$

$$= \sup_{c : \|c\|_2 \leq 1} \inf_{\nu \in M(A)} \sum_{i=1}^n c_i \, \nu(x_i \neq Y_i)$$

(by Sion's minimax Theorem) $$= \inf_{\nu \in M(A)} \sup_{c : \|c\|_2 \leq 1} \sum_{i=1}^n c_i \, \nu(x_i \neq Y_i)$$

(by CS) $$= \inf_{\nu \in M(A)} \sqrt{\sum_{i=1}^n \nu(x_i \neq Y_i)^2}$$

Let $$g_i(x^{(i)}) = \inf_{x_i' \in \mathcal{X}} g(x^{i-1}, x_i', x_{i+1}^n).$$

$*\quad \boxed{g(x) - g_i(x^{(i)}) \le 1}$

Let $g_i(x^{(i)}) = g(x^{i-1}, \tilde{x}_i, x_{i+1}^n)$

$$= \sum_{j \neq i} \tilde{\nu}(x_j \neq y_j)^2 + \tilde{\nu}(\tilde{x}_i \neq y_i)$$

Then, $g(x) \le \sum_{j=1}^{n} \tilde{\nu}(x_j \neq y_j)^2$

$$\Rightarrow g(x) - g_i(x^{(i)}) \le \tilde{\nu}(\tilde{x}_i \neq y_i) - \tilde{\nu}(x_i \neq y_i)$$

$$\le 1.$$

$*\quad \boxed{\sum_{i=1}^{n} \left(g(x) - g_i(x^{(i)})\right)^2 \le 4\, g(x)}$

It suffices to show:

$$\boxed{\begin{array}{l} \exists\, c \text{ s.t. } \|c\|_2 \le 1 \text{ and} \\ \left(\sqrt{g(x)} - \sqrt{g_i(x^{(i)})}\right)^2 \le c_i^2, \quad 1 \le i \le n \end{array}} \quad (\#)$$

Then,

$$\sum_{i=1}^{n} \left(g(x) - g_i(x^{(i)})\right)^2 \le \sum_{i=1}^{n} c_i^2 \cdot 4\, g(x)$$

$$\le 4\, g(x).$$

It remains to show (#). Let $\tilde{c}$ achieve $\sqrt{g(x)}$, i.e.,

$$\sqrt{g(x)} = \inf_{\nu \in \mathcal{M}(A)} \sum_{i=1}^{n} \tilde{c}_i\, \nu(x_i \neq y_i).$$

Further, noting that

$$\sqrt{g_i(x^{(i)})} \geq \inf_{\nu \in \mathcal{M}(A)} \sum_{j \neq i} \tilde{c}_j \, \nu(x_j \neq Y_j) + \tilde{c}_i \, \nu(\tilde{x}_i \neq Y_i)$$

$$= \sum_{j \neq i} \tilde{c}_j \, \tilde{\nu}(x_j \neq Y_j) + \tilde{c}_i \, \tilde{\nu}(\tilde{x}_i \neq Y_i).$$

Thus,
$$\sqrt{g(x)} - \sqrt{g_i(x^{(i)})} \leq \tilde{c}_i \left[\tilde{\nu}(x_i \neq Y_i) - \tilde{\nu}(\tilde{x}_i \neq Y_i) \right] \leq \tilde{c}_i,$$

and since $g(x) \geq g_i(x^{(i)})$,

$$\left(\sqrt{g(x)} - \sqrt{g_i(x^{(i)})} \right)^2 \leq \tilde{c}_i^2.$$

∎

F Examples

Ex1 Longest increasing subsequence

$f(x)$ = length of the longest increasing subsequence of x.

* f satisfies BDP with $(1, 1, \ldots, 1)$

$$\Rightarrow \mathbb{P}(|f(X) - \mathbb{E}f(X)| > t) \leq 2 \cdot e^{-2t^2/n}.$$

* $f(x)/\sqrt{n}$ is Lipchitz in d_T.

$$\Rightarrow \mathbb{P}(f(X) > \mathbb{M}f(X) + t) \leq 2 \cdot e^{-t^2/4n}.$$

Proof. Let $\mathcal{I}(x)$ denote the indices corresponding to the LIS of x. Then, $f(x) = |\mathcal{I}(x)|$ and so, 5

$$f(y) \geq \sum_{i=1}^{n} \mathbb{1}_{\{i \in g(x),\ x_i = y_i\}}$$

$$= f(x) - \sum_{i=1}^{n} \mathbb{1}_{\{i \in g(x),\ x_i \neq y_i\}}$$

$$\Rightarrow f(x) - f(y) \leq \sum_{i=1}^{n} \mathbb{1}_{\{i \in g(x)\}} \cdot \mathbb{1}_{\{x_i \neq y_i\}}$$

(It is possible to improve the tail bound to

$$\mathbb{P}\left(f(X) > \mathbb{M}f(X) + t\right) \leq \exp\left(-\frac{t^2}{4(\mathbb{M}f + t)}\right)$$

Fact: $\mathbb{M}f(X) = O(\sqrt{n})$ for $X_1, \dots, X_n$ iid unif $\{0,1\}$.)