Lecture 9

Recap: * $\mathrm{Var}(Z) \le \sum_{i=1}^{n} \mathbb{E}[\mathrm{Var}_{(i)}(Z)]$

$$\boxed{* \; q_k - q_{k-1} \le c \Rightarrow \mathbb{P}(Z > \mathbb{M}[Z] + t) < e^{-t/c}}$$

A Concentration around median (contd.)

Theorem (concentration around median)

Consider $Z = f(X_1, \dots, X_n)$ s.t.

$$\sum_{i=1}^{n} (Z - Z_i')_+^2 \le v,$$

where $Z_i' = f(X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_n)$ where X' is an indep. copy of X. Then,

$$\mathbb{P}(Z > \mathbb{M}[Z] + t) \le 2^{-t/\sqrt{8v}}.$$

Proof. By prop. (ii), it suffices to show that $q_{k+1} - q_k \le \sqrt{8v}$.

In fact, we show that for $0 < \delta < r \le \frac{1}{2}$,

$$(1) \qquad Q_{1-\delta} - Q_{1-r} \le \sqrt{4v \cdot \frac{r}{\delta}}.$$

The proof will be completed upon choosing $r = 2^{-k}, \delta = 2^{-k-1}$.

To see (1), for $b > a \ge \mathbb{M}Z$, let

$$Y = \begin{cases} b, & Z \ge b \\ Z, & b > Z > a \\ a, & Z \le a. \end{cases}$$

Then, $\mathbb{E}[Y] \le a \cdot \mathbb{P}(Z \le a) + b \cdot (1 - \mathbb{P}(Z \le a))$

$$= b - (b-a) \cdot \underbrace{\mathbb{P}(Z \le a)}_{\geq 1/2} \le \frac{a+b}{2}.$$

Thus,

$$\mathrm{Var}(Y) \geq \mathbb{P}(Y=b)\left(b-\mathbb{E}[Y]\right)^2$$

$$\geq \mathbb{P}(Y=b) \cdot \frac{(b-a)^2}{4}$$

$$= \mathbb{P}(Z \geq b)\,\frac{(b-a)^2}{4}.$$

On the other hand, by the Efron-Stein inequality,

$$\mathrm{Var}(Y) \leq \frac{1}{2}\sum_{i=1}^{n}\mathbb{E}\left[(Y-Y_i')^2\right]$$

$$= \sum_{i=1}^{n}\mathbb{E}\left[(Y-Y_i')_+^2\right]$$

$$= \sum_{i=1}^{n}\mathbb{E}\left[(Y-Y_i')_+^2 \cdot \mathbb{1}(Y>a)\right]$$

(since if $Y=a$, $Y_i' \geq Y$)

$$= \sum_{i=1}^{n}\mathbb{E}\left[(Y-Y_i')_+^2 \cdot \mathbb{1}(Z>a)\right]$$

$$\leq \sum_{i=1}^{n}\mathbb{E}\left[(Z-Z_i')_+^2\,\mathbb{1}(Z>a)\right]$$

(make cases and show)

$$\leq v\,\mathbb{P}(Z>a).$$

Combining the bounds above,

$$(b-a) \leq \sqrt{4v\,\frac{\mathbb{P}(Z>a)}{\mathbb{P}(Z\geq b)}}.$$

Choose $a = Q_{1-r}$ so that $\mathbb{P}(Z>a) \leq r$.

Choose $b < Q_{1-\delta}$ so that $\mathbb{P}(Z \geq b) \geq \delta$ since otherwise $\mathbb{P}(Z \leq b) \geq \mathbb{P}(Z<b) > 1-\delta$, which is a contradiction.

Thus, for every $b < Q_{1-\delta}$, $(b - Q_{1-r}) \leq \sqrt{4v\,r/\delta}$.

$$\Rightarrow Q_{1-\delta} - Q_{1-r} \leq \sqrt{4v\,r/\delta}. \quad \blacksquare$$

Concentration around mean using Efron-Stein

Recall how in the proof of Hoeffding lemma a bound for variance was used to prove a bound for $\psi_X(\lambda)$. In particular, the variance of X under a tilted distribution came into action. This provides an evidence for a connection b/w $\psi_X(\lambda)$ and variance. In the proof of the result below we present another such connection, which will be generalized later under the Herbst's argument and the entropy method.

Theorem (concentration around mean)

$Z = f(X)$, $X = X_1, \dots, X_n$, indep. $X_1, \dots, X_n$

$Z_i' = f(X^{i-1}, X_i', X_{i+1}^n)$, X' indep. copy of X

Suppose $\sum_{i=1}^n (Z - Z_i')_+^2 \leq v$. Then,

$$\mathbb{P}\left(Z > \mathbb{E}[Z] + t\right) \leq 2 . e^{-t/\sqrt{v}}.$$

Proof. Consider $Y = e^{\lambda Z/2}$. Then,

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}[e^{\lambda Z}] - \mathbb{E}[e^{\lambda Z/2}]^2 \\
&\leq \sum_{i=1}^n \mathbb{E}\left[(Y - Y_i')_+^2\right]
\end{aligned}$$

Note that $Z \geq Z_i \Leftrightarrow e^{\lambda Z/2} \geq e^{\lambda Z_i/2}$.

Further, $e^{\lambda x/2} \leq e^{\lambda y/2} + \frac{\lambda}{2}(x-y)e^{\frac{\lambda\theta}{2}}$, $\theta \in [y, x]$.

Therefore, $(Y-Y_i')_+^2 \le \frac{\lambda^2}{4}(Z-Z_i')_+^2 \, e^{\lambda Z}$.

which gives

$$\mathrm{Var}(Y) \le \frac{\lambda^2}{4} \mathbb{E}\left[\sum_{i=1}^n (Z-Z_i')_+^2 \, e^{\lambda Z}\right]$$

$$\le \frac{\lambda^2}{4} \cdot v \cdot \mathbb{E}\left[e^{\lambda Z}\right]$$

Thus,

$$\mathbb{E}\left[e^{\lambda(Z-\mathbb{E}Z)}\right] - \mathbb{E}\left[e^{\frac{\lambda}{2}(Z-\mathbb{E}Z)}\right]^2$$

$$\le \frac{\lambda^2 v}{4} \cdot \mathbb{E}\left[e^{\lambda(Z-\mathbb{E}Z)}\right],$$

i.e.,

$$\left(1-\frac{\lambda^2 v}{4}\right) \cdot \mathbb{E}\left[e^{\lambda(Z-\mathbb{E}Z)}\right] \le \mathbb{E}\left[e^{\frac{\lambda}{2}(Z-\mathbb{E}Z)}\right]^2$$

Assuming $1 \ge \frac{\lambda^2}{4} v$ and taking log

$$\psi_{Z-\mathbb{E}Z}(\lambda) + \log\left(1-\frac{\lambda^2}{4}v\right) \le 2\,\psi_{Z-\mathbb{E}Z}\left(\frac{\lambda}{2}\right).$$

A functional inequality

$$f(\lambda) \le -\log\left(1-\frac{\lambda^2}{4}v\right) + 2f\left(\frac{\lambda}{2}\right), \quad 0 \le \lambda \le \frac{2}{\sqrt{v}}$$

Thus,

$$f(\lambda) \le -\sum_{i=1}^k \log\left(1-\frac{\lambda^2 v}{4\,2^{2i}}\right) \cdot 2^i + 2^{k+1} \cdot f\left(\frac{\lambda}{2^{k+1}}\right).$$

Taking limit $k\to\infty$, since

$$\lim_{k\to\infty} 2^{k+1} f\left(\frac{\lambda}{2^{k+1}}\right) = \lambda \lim_{k\to\infty} \frac{f\left(\frac{\lambda}{2^{k+1}}\right)}{\lambda/2^{k+1}} = \lambda f'(0)$$

$$= 0,$$

$-\frac{1}{u}\log(1-xu) \le -\log(1-x)$

we get

$$f(\lambda) \le \sum_{i=1}^k 2^i \log\left(\frac{1}{1-\frac{\lambda^2 v}{4\,2^{2i}}}\right) \le -2\log\left(1-\frac{\lambda^2 v}{4}\right).$$

In particular,
$$f\left(\frac{1}{\sqrt{v}}\right) \leq \log \frac{16}{9}$$
Thus,
$$\mathbb{P}(Z - \mathbb{E}[Z] > t) \leq e^{-t/\sqrt{v}} \cdot e^{f\left(\frac{1}{\sqrt{v}}\right)}$$
$$\leq 2 \cdot e^{-t/\sqrt{v}} \qquad \blacksquare$$

B The Gaussian Poincaré inequality

Theorem. $X = (X_1, \dots, X_n)$, X_i indep., $X_i \sim \mathcal{N}(0,1)$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable func. s.t. $\left|\frac{\partial^2 f}{\partial x_i^2}\right|$ and $\left|\frac{\partial f}{\partial x_i}\right|$ is bounded for every i. Then,
$$\boxed{\operatorname{Var}(f(X)) \leq \mathbb{E}\left[\|\nabla f(X)\|^2\right]}$$

Proof. It suffices to show the bound for $n=1$. Indeed,
$$\operatorname{Var}(f(x)) \leq \sum_{i=1}^n \mathbb{E}\left[\operatorname{Var}_{(i)}(f(x))\right]$$
(using the ineq. for $n=1$)
$$\leq \sum_{i=1}^n \mathbb{E}\left[\mathbb{E}_{(i)}\left[\left(\frac{\partial}{\partial x_i} f(x)\right)^2\right]\right]$$
$$= \mathbb{E}\left[\|\nabla f(x)\|^2\right]$$

We now show the inequality for $n=1$. Consider iid Radamacher $\epsilon_1, \dots, \epsilon_m$. Let $S_m = \frac{1}{\sqrt{m}} \sum_{i=1}^m \epsilon_i$.

Consider
$$g(\epsilon_1, \dots, \epsilon_m) = f(S_m).$$

Therefore, $\mathrm{Var}_{(i)}\left(g(\epsilon)\right)$

$$= \frac{1}{4}\left(g(\epsilon_1,\ldots,\epsilon_{i-1},1,\epsilon_{i+1},\ldots,\epsilon_m) - g(\epsilon_1,\ldots,\epsilon_{i-1},-1,\epsilon_{i+1},\ldots,\epsilon_m)\right)^2$$

$$= \frac{1}{4}\left(f\left(S_m - \frac{\epsilon_i}{\sqrt{m}} + \frac{1}{\sqrt{m}}\right) - f\left(S_m - \frac{\epsilon_i}{\sqrt{m}} - \frac{1}{\sqrt{m}}\right)\right)^2$$

$$\leq \frac{1}{4}\left(\frac{2}{\sqrt{m}}\left|f'(S_m)\right| + \frac{2}{m}K\right)^2$$

$$= \frac{1}{n}\left(f'(S_m)^2 + \frac{2K f'(S_m)}{\sqrt{m}} + \frac{K^2}{m}\right).$$

Thus,

$$\mathrm{Var}\left(f(S_m)\right) = \mathrm{Var}\left(g(\epsilon)\right)$$

$$\leq \mathbb{E}\left[f'(S_m)^2\right] + \frac{2K}{\sqrt{m}}\mathbb{E}\left[f'(S_m)\right] + \frac{K^2}{m}$$

Taking limit $m \to \infty$ and using the central limit theorem,

$$\mathrm{Var}\left(f(X)\right) \leq \mathbb{E}\left[f(X)^2\right].$$

∎

This inequality provides an instance of Talagrand's principle.

The recipe above is generic :

* Use tensorization to replace n with 1
* Get Gaussian result as a limit for discrete