

Secret Key Agreement: General Capacity and Second-Order Asymptotics

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Abstract

We revisit the problem of secret key agreement using interactive public communication for two parties and propose a new secret key agreement protocol. The protocol attains the secret key capacity for general observations and attains the second-order asymptotic term in the maximum length of a secret key for independent and identically distributed observations. In contrast to the previously suggested secret key agreement protocols, the proposed protocol uses interactive communication. In fact, the standard one-way communication protocol used prior to this work fails to attain the asymptotic results above. Our converse proofs rely on a recently established upper bound for secret key lengths. Both our lower and upper bounds are derived in a *single-shot* setup and the asymptotic results are obtained as corollaries.

I. INTRODUCTION

Two parties observing *random variables* (RVs) X and Y seek to agree on a secret key. They can communicate interactively over an error-free, authenticated, albeit insecure, communication channel of unlimited capacity. The secret key must be concealed from an eavesdropper with access to the communication and an additional side information Z . What is the maximum length $S(X, Y | Z)$ of a secret key that the parties can agree upon?

A study of this question was initiated by Maurer [19] and Ahlswede and Csiszár [1] for the case where the observations of the parties and the eavesdropper consist of *n independent and identically distributed*

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(IID) repetitions (X^n, Y^n, Z^n) of RVs (X, Y, Z) . For the case when $X \ominus Y \ominus Z$ form a Markov chain, it was shown in [19], [1] that the *secret key capacity* equals $I(X \wedge Y | Z)$, namely

$$S(X^n, Y^n | Z^n) = nI(X \wedge Y | Z) + o(n).$$

However, in several applications (see, for instance, [6]) the observed data is not IID or even if the observations are IID, the observation length n is limited and a more precise asymptotic analysis is needed.

In this paper, we address the secret key agreement problem for these two important practical situations. First, when the observations consist of general sources (*cf.* [11], [10]) (X_n, Y_n, Z_n) such that $X_n \ominus Y_n \ominus Z_n$ is a Markov chain, we show that

$$S(X_n, Y_n | Z_n) = n\underline{I}(\mathbf{X} \wedge \mathbf{Y} | \mathbf{Z}) + o(n),$$

where $\underline{I}(\mathbf{X} \wedge \mathbf{Y} | \mathbf{Z})$ is the *inf-conditional information* of \mathbf{X} and \mathbf{Y} given \mathbf{Z} . Next, for the IID case with $X \ominus Y \ominus Z$, we identify the second-order asymptotic term¹ in $S(X^n, Y^n | Z^n)$. Specifically, denoting by $S_{\epsilon, \delta}(X, Y | Z)$ the maximum length of a secret key over which the parties agree with probability greater than $1 - \epsilon$ and with secrecy parameter less than δ , we show that

$$S_{\epsilon, \delta}(X^n, Y^n | Z^n) = nI(X \wedge Y | Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta) \pm \mathcal{O}(\log n),$$

where Q is the tail probability of the standard Gaussian distribution and

$$V := \mathbb{V}\text{ar} \left[\log \frac{P_{XY|Z}(X, Y | Z)}{P_{X|Z}(X | Z) P_{Y|Z}(Y | Z)} \right].$$

In particular, our bounds allow us to evaluate the *gap to secret key capacity* at a finite blocklength n . In Figure 1 we illustrate this gap between the maximum possible rate of a secret key at a fixed n and the secret key capacity for the case where Z is a random bit, Y is obtained by flipping Z with probability 0.25 and X given by flipping Y with probability 0.125; see Example 1 in Section VI for details.

Underlying these results is a general single-shot characterization of the secret key length which shows that, when $X \ominus Y \ominus Z$, $S_{\epsilon, \delta}(X, Y | Z)$ roughly equals the $(\epsilon + \delta)$ -tail of the random variable

$$i(X \wedge Y | Z) = \log \frac{P_{XY|Z}(X, Y | Z)}{P_{X|Z}(X | Z) P_{Y|Z}(Y | Z)}.$$

Our main technical contribution in proving this result is a new single-shot secret key agreement protocol

¹Following the pioneering work of Strassen [28], study of these second-order terms in coding theorems has been revived recently by Hayashi [13], [14] and Polyanskiy, Poor, and Verdú [23].

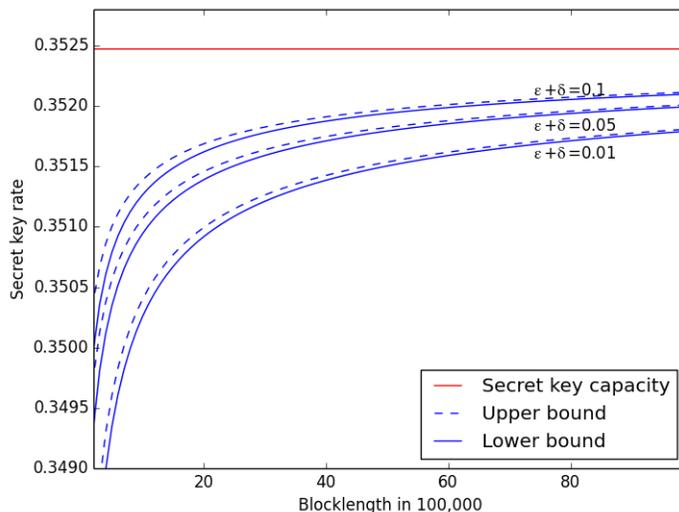


Fig. 1. Gap to secret key capacity at finite n for $\epsilon + \delta = 0.01, 0.05, 0.1$.

which uses interactive communication and attains the desired optimal performance. It was observed in [19], [1] that a simple one-way communication protocol suffices to attain the secret key capacity, when the Markov relation $X \leftrightarrow Y \leftrightarrow Z$ holds. Also, for a multiterminal setup with constant Z , [4] showed that a noninteractive communication protocol achieves the secret key capacity. Prior to this work, when the Markov relation $X \leftrightarrow Y \leftrightarrow Z$ holds, only such noninteractive communication protocols were used for generating secret keys², even in single-shot setups (*cf.* [25]). In contrast, our proposed protocol uses interactive communication. We note in Remark 4 that none of the standard one-way communication protocols achieve the optimal asymptotic bounds, suggesting that perhaps interaction is necessary for generating a secret key of optimal length (see Section VII for further discussion and an illustrative example).

Typically, secret key agreement protocols consist of two steps: *information reconciliation* and *privacy amplification*. In the first step, the parties communicate to generate some shared random bits, termed *common randomness*. However, the communication used leaks some information about the generated common randomness. Therefore, a second privacy amplification step is employed to extract from the common randomness secure random bits that are almost independent of the communication used. For IID observations, the information reconciliation step of the standard one-way secret key agreement protocol entails the two parties agreeing on X using a one-way communication of rate $H(X|Y)$. In the privacy

²Interaction is known to help in some cases where neither $X \leftrightarrow Y \leftrightarrow Z$ nor $Y \leftrightarrow X \leftrightarrow Z$ is satisfied [19], [35], [9].

amplification step, the rate $H(X|Z)$ residual randomness of X , which is almost independent of Z , is used to extract a secret key of rate $H(X|Z) - H(X|Y)$ which is independent jointly of Z and the communication used in the information reconciliation stage. Under the Markov condition $X \circlearrowleft Y \circlearrowleft Z$, the resulting secret key attains the secret key capacity $I(X \wedge Y | Z)$. However, in the single-shot regime, the behavior of RVs $-\log P_{X|Z}(X|Z)$ and $-\log P_{X|Y}(X|Y)$, rather than their expected values, becomes relevant. The difficulty in extending the standard one-way communication protocol to the single-shot setup lies in the spread of the *information spectrums*³ of $P_{X|Y}$ and $P_{X|Z}$. Specifically, while we require the random variable $-\log P_{X|Y}(X|Y)$ itself to show up as the length of communication in the information reconciliation step, a naive extension requires as much communication as a large probability tail of $-\log P_{X|Y}(X|Y)$. To remedy this, we *slice the spectrum* of $P_{X|Y}$ into slices⁴ of length Δ each and adapt the protocol to the slice which contains (X, Y) . However, since neither party knows the value of $-\log P_{X|Y}(X|Y)$, this adaptation requires interactive communication.

Motivating this work, and underlying our converse proof, is a recently established single-shot upper bound on secret key lengths for the multiparty secret key agreement problem [33] (see, also, [31]). The proof relies on relating secret key agreement to binary hypothesis testing. In spirit, this result can be regarded as a multiterminal variant of a similar single-shot converse for the channel coding problem which appeared first in [21], [15] and has been termed the meta-converse by Polyanskiy, Poor, and Verdú [23], [22] (see, also, [34] and [12, Section 4.6]).

The basic concepts of secret key agreement and a general result for converting a high reliability protocol to a high secrecy protocol are given in the next section. In Section III, we review the single-shot upper bound of [33] for the two party case. Our new secret key agreement protocol and its single-shot performance analysis is presented in Section IV. The single-shot results are applied to general sources in Section V and to IID sources in Section VI. The final section contains a discussion on the role of interaction in our secret key agreement protocols.

II. SECRET KEYS

We consider the problem of secret key agreement using interactive public communication by two (trusted) parties observing RVs X and Y taking values in countable sets \mathcal{X} and \mathcal{Y} , respectively. Upon making these observations, the parties communicate interactively over a public communication channel

³The range of the log-likelihood $-\log P_X(x)$ (conditional log-likelihood $-\log P_{X|Y}(x|y)$) is referred to as information spectrum of P_X (conditional information spectrum of $P_{X|Y}$). This notion was introduced in the seminal work [11] and is appropriate for deriving single-shot coding theorems, without making assumptions on the underlying distribution. See [10] for a detailed account.

⁴See Appendix C for a secret key agreement based on slicing the spectrum of P_X .

that is accessible by an eavesdropper. We assume that the communication channel is error-free and authenticated. Specifically, the communication is sent over r rounds of interaction⁵. In the j th round of communication, $1 \leq j \leq r$, each party sends a message which is a function of its observation, *locally generated* randomness denoted by⁶ U_x and U_y , and the previously observed communication. The overall interactive communication is denoted by \mathbf{F} . In addition to \mathbf{F} , the eavesdropper observes a RV Z taking values in a countable set \mathcal{Z} . The joint distribution P_{XYZ} is known to all parties.

Using the interactive communication \mathbf{F} and their local observations, the parties agree on a secret key. A RV K constitutes a secret key if the two parties form estimates that agree with K with probability close to 1 and K is concealed, in effect, from an eavesdropper with access to (\mathbf{F}, Z) . Formally, we have the following definition.

Definition 1. A RV K with range \mathcal{K} constitutes an (ϵ, δ) -secret key $((\epsilon, \delta)$ -SK) if there exist functions K_x and K_y of (U_x, X, \mathbf{F}) and (U_y, Y, \mathbf{F}) , respectively, such that the following two conditions are satisfied

$$P(K_x = K_y = K) \geq 1 - \epsilon, \quad (1)$$

$$\|P_{K\mathbf{F}Z} - P_{\text{unif}} \times P_{\mathbf{F}Z}\|_1 \leq \delta, \quad (2)$$

where P_{unif} is the uniform distribution on \mathcal{K} and

$$\|P - Q\|_1 = \frac{1}{2} \sum_u |P(u) - Q(u)|.$$

The first condition above represents the *reliability* of the secret key and the second condition guarantees *secrecy*.

Definition 2. Given $\epsilon, \delta \in [0, 1)$, the supremum over the lengths $\log |\mathcal{K}|$ of an (ϵ, δ) -SK is denoted by $S_{\epsilon, \delta}(X, Y | Z)$.

Remark 1. The only interesting case is when $\epsilon + \delta < 1$, since otherwise $S_{\epsilon, \delta}(X, Y | Z)$ is unbounded. Indeed, consider two trivial secret keys K_1 and K_2 with range \mathcal{K} generated as follows: For K_1 , the first party generates $K_x = K_1$ uniformly over \mathcal{K} and sends it to the second party. Thus, K_1 constitutes a $(0, 1 - 1/|\mathcal{K}|)$ -SK, and therefore, also a $(0, 1)$ -SK. For K_2 , the first party generates $K_x = K_2$ uniformly over \mathcal{K} and the second party generates K_y uniformly over \mathcal{K} . Then, K_2 constitutes a $(1 - 1/|\mathcal{K}|, 0)$ -SK, and therefore, also a $(1, 0)$ -SK. If $\epsilon + \delta \geq 1$, the RV K which equals K_1 with probability $(1 - \epsilon)$ and

⁵In the asymptotic regime considered in Sections V, the number of rounds r may depend on the block length n .

⁶The RVs U_x and U_y are mutually independent and independent jointly of (X, Y) .

K_2 with probability ϵ constitutes $(\epsilon, 1 - \epsilon)$ -SK of length $\log |\mathcal{K}|$, and therefore, also an (ϵ, δ) -SK of the same length. Since \mathcal{K} was arbitrary, $S_{\epsilon, \delta}(X, Y | Z) = \infty$.

Remark 1 exhibits a high reliability $(0, 1)$ -SK and a high secrecy $(1, 0)$ -SK for the trivial case $\epsilon + \delta \geq 1$. The two constructions together sufficed to characterize $S_{\epsilon, \delta}(X, Y | Z)$. Following a similar approach for the regime $\epsilon + \delta < 1$, we can construct a high reliability $(\epsilon + \delta, 0)$ -SK and a high secrecy $(0, \epsilon + \delta)$ -SK and randomize over those two secret keys with probabilities $\epsilon/(\epsilon + \delta)$ and $\delta/(\epsilon + \delta)$ to obtain a hybrid, (ϵ, δ) -SK. However, the results below show that we do not need to construct both high secrecy and high reliability secret keys for the secrecy definition in (2) and a high reliability construction alone will suffice. We first show that any (ϵ, δ) -SK can be converted into a high secrecy, $(\epsilon + \delta, 0)$ -SK.

Proposition 1 (Conversion to High Secrecy Protocol). *Given an (ϵ, δ) -SK, there exists an $(\epsilon + \delta, 0)$ -SK of the same length.*

Proof. Let K be an (ϵ, δ) -SK using interactive communication \mathbf{F} , with local estimates K_x and K_y . We construct a new $(\epsilon + \delta, 0)$ -SK K' using the *maximal coupling lemma*, which asserts the following (cf. [29]): Given two distributions P and Q on a set \mathcal{X} , there exists a joint distribution $P_{XX'}$ on $\mathcal{X} \times \mathcal{X}$ such that the marginals are $P_X = P$ and $P_{X'} = Q$, and under $P_{XX'}$

$$P(X \neq X') = \|P - Q\|_1. \quad (3)$$

The distribution $P_{XX'}$ is called the *maximal coupling* of P and Q .

For each fixed realization of (\mathbf{F}, Z) , let $P_{K'K'|\mathbf{F}, Z}$ be the maximal coupling of $P_{K|\mathbf{F}, Z}$ and P_{unif} . Then $P_{K'\mathbf{F}, Z} = P_{\text{unif}} \times P_{\mathbf{F}, Z}$, and since K is an (ϵ, δ) -SK, we get by the maximal coupling property (3) that

$$P(K \neq K') = \|P_{K\mathbf{F}, Z} - P_{\text{unif}} \times P_{\mathbf{F}, Z}\|_1 \leq \delta,$$

and define

$$P_{K'K_x K_y \mathbf{F} X Y Z U_x U_y} := P_{K'|\mathbf{F}, Z} P_{K_x K_y \mathbf{F} X Y Z U_x U_y}. \quad (4)$$

Since $P(K = K_x = K_y) \geq 1 - \epsilon$, under $P_{K'K_x K_y \mathbf{F} X Y Z}$ we have

$$P(K_x = K_y = K') \geq 1 - \epsilon - \delta.$$

Thus, K' constitutes an $(\epsilon + \delta, 0)$ -SK. \square

Proposition 1 plays an important role in our secret key agreement protocol and allows us to convert a high reliability (η, α) -SK with small η into an (ϵ, δ) -SK for any arbitrary ϵ and δ satisfying (roughly)

$\epsilon + \delta > \alpha$. Formally, we have the following.

Proposition 2 (Hybrid Protocol). *Given a protocol for generating (η, α) -SK, there exists a protocol for generating an (ϵ, δ) -SK of the same length for every $0 < \epsilon, \delta < 1$ such that*

$$\begin{aligned} \epsilon &\geq \eta \\ \epsilon + \delta &\geq \alpha + \eta. \end{aligned}$$

Proof. Given an (η, α) -SK K_1 , by Proposition 1 there exists an $(\alpha + \eta, 0)$ -SK K_2 . Let $\theta = \delta / (\epsilon - \eta + \delta)$. Consider a secret key K obtained by a hybrid use of the protocols for generating K_1 and K_2 , with the protocol for K_1 executed with probability θ and that for K_2 with probability $1 - \theta$. Note from the proof of Proposition 1 that it is the same secret key agreement protocol (K_x, K_y, \mathbf{F}) that generates both K_1 and K_2 . Thus, the claim follows for the time-shared secret key K since

$$\begin{aligned} \mathbb{P}(K = K_x = K_y) &= \theta \mathbb{P}(K_1 = K_x = K_y) + (1 - \theta) \mathbb{P}(K_2 = K_x = K_y) \\ &\geq 1 - \frac{\delta \eta + (\epsilon - \eta)(\alpha + \eta)}{\epsilon - \eta + \delta} \\ &\geq 1 - \frac{\delta \eta + (\epsilon - \eta)(\epsilon + \delta)}{\epsilon - \eta + \delta} \\ &= 1 - \epsilon, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \|\mathbb{P}_{K\mathbf{F}Z} - \mathbb{P}_{\text{unif}}\mathbb{P}_{\mathbf{F}Z}\|_1 &\leq \theta \|\mathbb{P}_{K_1\mathbf{F}Z} - \mathbb{P}_{\text{unif}}\mathbb{P}_{\mathbf{F}Z}\|_1 + (1 - \theta) \|\mathbb{P}_{K_2\mathbf{F}Z} - \mathbb{P}_{\text{unif}}\mathbb{P}_{\mathbf{F}Z}\|_1 \\ &\leq \frac{\delta \alpha + (\epsilon - \eta) \cdot 0}{\epsilon - \eta + \delta} \\ &\leq \frac{\delta(\epsilon + \delta - \eta)}{\epsilon - \eta + \delta} \\ &= \delta, \end{aligned} \tag{6}$$

where we have used the assumption $\epsilon + \delta \geq \alpha + \eta$ in (5) and (6). \square

Remark 2. Note that the actual secret key K in Definition 1 is not available to any party and has only a formal role in the secret key agreement protocol. Interestingly, the proof above says that the estimates (K_x, K_y) of a high reliability (η, α) -SK with $\eta \approx 0$ constitute an (ϵ, δ) -SK as well for every $\epsilon + \delta \gtrsim \alpha$, albeit for a different hidden RVs K .

Thus, it suffices to exhibit a high reliability (η, α) -SK with small η and desired α . Such a protocol is

given in Section IV and underlies all our achievability results.

A more demanding secrecy requirement. A more demanding secrecy requirement enforces one of the estimates K_x or K_y itself to be secure, *i.e.*,

$$\|P_{K_x \mathbf{F}Z} - P_{\text{unif}} P_{\mathbf{F}Z}\|_1 \leq \delta \text{ or } \|P_{K_y \mathbf{F}Z} - P_{\text{unif}} P_{\mathbf{F}Z}\|_1 \leq \delta. \quad (7)$$

The validity of Proposition 1 for this secrecy requirement remains open. However, in the important special case when Z is a function of either X or Y , which includes the case of constant Z , Proposition 1 holds even under the more demanding secrecy requirement (7). Indeed, let (K_x, K_y) be an (ϵ, δ) -SK with K_x satisfying the more demanding secrecy requirement above. Proceeding as in the proof of Proposition 1 with K_x in the role of K , we obtain a RV K' such that $P(K' \neq K_y) \leq \epsilon + \delta$ and $P_{K' \mathbf{F}Z} = P_{\text{unif}} \times P_{\mathbf{F}Z}$. Let the joint distribution $P_{K'K_xK_y \mathbf{F}XYZU_xU_y}$ be as in (4) with K_x replacing K . To claim that K' constitutes an $(\epsilon + \delta, 0)$ -SK under (7), it suffices to show that one of the parties can simulate K' . To that end, the party observing X can first run the original secret key agreement protocol to get K_x and \mathbf{F} . Also, Z is available to the party since it is a function of X . Thus, this party can simulate the required RV K' using the distribution $P_{K'|K_x \mathbf{F}Z}$, which completes the proof of Proposition 1 under the more demanding secrecy requirement.

Note that the argument above relies on using local randomness to simulate K' . For the original secrecy requirement (2), Proposition 1 holds even when we restrict to deterministic protocols with no local randomness allowed. It turns out that this is not the case for the more demanding secrecy requirement (7), as the following simple counterexample shows: Let X be a binary RV taking 1 with probability $p < \frac{1}{2}$, and let $Y = Z = \text{constant}$. Then, $K_x = X$ and constitutes a 1-bit $(p, 1/2 - p)$ -SK under (7). If Proposition 1 holds, the parties should be able to generate a $(1/2, 0)$ -SK. However, a $(1/2, 0)$ -SK consists of an unbiased bit, which cannot be generated without additional randomness. Therefore, for secrecy requirement (7), Proposition 1 does not hold if we restrict to deterministic protocols.

To conclude, for the special case when Z is a function of X , it suffices to construct only a high reliability protocol, provided that local randomness is available. In fact, the high reliability protocol proposed in Section IV satisfies the more demanding secrecy requirement (7) and, if Z is a function of either X or Y , all the results of this paper hold even under (7).

III. UPPER BOUND ON $S_{\epsilon, \delta}(X, Y | Z)$

We recall the *conditional independence testing* upper bound on $S_{\epsilon, \delta}(X, Y | Z)$, which was established recently in [33], [32]. In fact, the general upper bound in [33], [32] is a single-shot upper bound on the secret key length for a multiparty secret key agreement problem. We recall a specialization of the

general result to the case at hand. In order to state our result, we need the following concept from binary hypothesis testing.

Consider a binary hypothesis testing problem with null hypothesis P and alternative hypothesis Q , where P and Q are distributions on the same alphabet \mathcal{V} . Upon observing a value $v \in \mathcal{V}$, the observer needs to decide if the value was generated by the distribution P or the distribution Q . To this end, the observer applies a stochastic test T , which is a conditional distribution on $\{0, 1\}$ given an observation $v \in \mathcal{V}$. When $v \in \mathcal{V}$ is observed, the test T chooses the null hypothesis with probability $T(0|v)$ and the alternative hypothesis with probability $T(1|v) = 1 - T(0|v)$. For $0 \leq \epsilon < 1$, denote by $\beta_\epsilon(P, Q)$ the infimum of the probability of error of type II given that the probability of error of type I is less than ϵ , *i.e.*,

$$\beta_\epsilon(P, Q) := \inf_{T: P[T] \geq 1-\epsilon} Q[T], \quad (8)$$

where

$$\begin{aligned} P[T] &= \sum_v P(v)T(0|v), \\ Q[T] &= \sum_v Q(v)T(0|v). \end{aligned}$$

The definition of a secret key used in [33], [32] is different from Definition 1. However, the two definitions are closely related, and the upper bound of [33], [32] can be extended to our case as well. We review the alternative definition in Appendix A and relate it to Definition 1 to derive the following upper bound, which will be instrumental in our converse proofs.

Theorem 3 (Conditional independence testing bound). *Given $0 \leq \epsilon + \delta < 1$, $0 < \eta < 1 - \epsilon - \delta$, the following bound holds:*

$$S_{\epsilon, \delta}(X, Y | Z) \leq -\log \beta_{\epsilon+\delta+\eta}(P_{XYZ}, Q_{X|Z}Q_{Y|Z}Q_Z) + 2\log(1/\eta),$$

for all joint distributions Q on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ that render X and Y conditionally independent given Z .

IV. THE SECRET KEY AGREEMENT PROTOCOL

In this section we present our secret key agreement protocol, which will be used in all the achievability results of this paper. A typical secret key agreement protocol for two parties entails sharing the observations of one of the parties, referred to as *information reconciliation*, and then extracting a secret key out of the shared observations, referred to as *privacy amplification* (*cf.* [19], [1], [24]). In another interpretation, the parties communicate first to establish a *common randomness* [2] and then extract a

secret key from the common randomness⁷. Our protocol below, too, has these two components but the rate of the communication for information reconciliation and the rate of the randomness extracted by privacy amplification have to be chosen carefully.

Heuristically, in the information reconciliation stage, the first party randomly bins X and sends it to the second party. If we do not use interaction, by the Slepian-Wolf theorem [27] (see [20],[10, Lemma 7.2.1], [18] for a single-shot version) the length of communication needed is roughly equals a large probability upper bound for $h(X|Y) := -\log P_{X|Y}(X|Y)$. However, in order to derive a lower bound that matches our upper bound, we expect to send X to Y using approximately $h(X|Y)$ bits of communication, which can differ from the tail bound above by as much as the length of the spectrum of $P_{X|Y}$. To overcome this gap, we utilize spectrum slicing, a technique introduced in [10], to construct an adaptive scheme that can handle the *spread of information spectrum*. Specifically, we divide the spectrum of $P_{X|Y}$ into slices of small lengths. The protocol proceeds interactively to adapt to the current slice index, allowing us to replace the spectrum length in the argument above with the length of a single slice.

A. Formal description of the protocol

We consider the essential spectrum of $P_{X|Y}$, *i.e.*, the set of values taken by $h(X|Y)$ between λ_{\min} and λ_{\max} , where λ_{\min} and λ_{\max} are chosen such that $h(X|Y)$ lies in $(\lambda_{\min}, \lambda_{\max})$ with large probability. We divide the essential spectrum of $P_{X|Y}$ into L slices, L of them of width Δ . Specifically, for $1 \leq j \leq L$, the j th slice of the spectrum of $P_{X|Y}$ is defined as follows

$$\mathcal{T}_j = \{(x, y) : \lambda_j \leq -\log P_{X|Y}(x|y) < \lambda_j + \Delta\},$$

where $\lambda_j = \lambda_{\min} + (j - 1)\Delta$. Note that the slice index is not available to any one party. The proposed protocol proceeds assuming the lowest index $j = 1$ and uses interactive communication to adapt to the actual slice index.

For information reconciliation, we simply send a random binning of X . However, the bin size is increased successively, where the incremental bin sizes M_1, \dots, M_L are given by

$$\log M_j = \begin{cases} \lambda_1 + \Delta + \gamma, & j = 1 \\ \Delta, & 1 < j \leq L, \end{cases}$$

⁷For an interpretation of secrecy agreement in terms of common randomness decomposition, see [4], [5], [30].

For privacy amplification, we will rely on the *leftover hash lemma* [17], [25]. Let \mathcal{F} be a *2-universal family* of mappings $f : \mathcal{X} \rightarrow \mathcal{K}$, i.e., for each $x' \neq x$, the family \mathcal{F} satisfies

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \mathbb{1}(f(x) = f(x')) \leq \frac{1}{|\mathcal{K}|}. \quad (9)$$

The following lemma is a slight modification of the known forms of the leftover hash lemma (cf. [24]); we give a proof in Appendix B for completeness.

Lemma 4 (Leftover Hash). *Consider RVs X , Z , and V taking values in \mathcal{X} , \mathcal{Z} and \mathcal{V} , respectively, where \mathcal{X} and \mathcal{Z} are countable and \mathcal{V} is finite. Let S be a random seed such that f_S is uniformly distributed over a 2-universal family as above. Then, for $K = f_S(X)$ and for any Q_Z satisfying $\text{supp}(P_Z) \subset \text{supp}(Q_Z)$, we have*

$$\|P_{K V Z S} - P_{\text{unif}} P_{V Z} P_S\|_1 \leq \frac{1}{2} \sqrt{|\mathcal{K}| |\mathcal{V}| 2^{-H_{\min}(P_{XZ} | Q_Z)}},$$

where P_{unif} is the uniform distribution on \mathcal{K} and

$$H_{\min}(P_{XZ} | Q_Z) = -\log \sup_{x, z: Q_Z(z) > 0} \frac{P_{XZ}(x, z)}{Q_Z(z)}.$$

The main benefit of the spectrum slicing approach above is that roughly $h(X|Y) + L + \Delta$ bits are sent for each realization (X, Y) . At the same time, we can estimate $h(X|Y)$ up to a precision of Δ when the protocol stops – a key property in our secrecy analysis.

The complete protocol is described in Protocol 1. We remark that the random seed based secret key generation used in Protocol 1 is only for the ease of security analysis. A slight modification of our protocol can work with deterministic extractors.

Protocol 1: Secret key agreement protocol

Input: Observations X and Y

Output: Secret key estimates K_x and K_y

Information reconciliation

Initiate the protocol with $l = 1$

while $l \leq L$ and ACK not received **do**

 First party sends the random bin index of X into M_l bins, $B_l = F_{1l}(X)$, to the second party

if Second party find a unique x such that $(x, Y) \in \mathcal{T}_l$ and $F_{1j}(x) = B_j, \quad \forall 1 \leq j \leq l$ **then**

 Second party sets $\hat{X} = x$ and sends back an ACK $F_{2i} = 1$ to the first party

else

 Second party sends back a NACK $F_{2i} = 0$

 Parties update $l \rightarrow l + 1$

if No ACK received **then**

 Protocol declares an error and aborts

else

Privacy amplification

 First party generates the random seed S and sends it to the second party using public communication

 First party generates the secret key $K_x = K = f_S(X)$

 The second party generates the estimate K_y of K as $K_y = f_S(\hat{X})$

Remark 3. Note that since each ACK-NACK signal will require 1-bit of communication to implement, the number of bits physically sent in the protocol is roughly $h(X|Y) + \Delta + L$. However, the log of the number of values taken by the transcript is much less, roughly $h(X|Y) + \Delta + \log L$, since the ACK-NACK sequence will be a stopped sequence consisting of NACKs followed by a single ACK. By Lemma 4, it is this latter quantity $h(X|Y) + \Delta + \log L$ that captures the amount of information leaked to the eavesdropper by public communication, which will be used in our security analysis of the protocol.

B. Performance analysis of the protocol

We now derive performance guarantees for the secret key agreement protocol of the previous section. In view of Proposition 2 and Remark 2, the protocol above will constitute an (ϵ, δ) -SK protocol for arbitrary $\epsilon, \delta \in (0, 1)$ if it yields an $(\eta, \epsilon + \delta)$ -SK with $\eta \approx 0$. The result below shows that Protocol 1 indeed yields such a high reliability secret key.

Denote by $i_{XY}(x, y)$ the information density

$$i_{XY}(x, y) := \log \frac{P_{XY}(x, y)}{P_X(x) P_Y(y)}.$$

Theorem 5. For $\lambda_{\min}, \lambda_{\max}, \Delta > 0$ with $\lambda_{\max} \geq \lambda_{\min}$, let

$$L = \frac{\lambda_{\max} - \lambda_{\min}}{\Delta}.$$

Then, for every $\gamma > 0$ and $\lambda \geq 0$, there exists an (ϵ, δ) -SK K taking values in \mathcal{K} with

$$\begin{aligned} \epsilon &\leq P_{XY}(\mathcal{T}_0) + L2^{-\gamma}, \\ \delta &\leq P(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda + \Delta) \\ &\quad + \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - \gamma - 3 \log L)}} + \frac{1}{L} + P_{XY}(\mathcal{T}_0) + L2^{-\gamma}, \end{aligned}$$

where

$$\mathcal{T}_0 := \{(x, y) : -\log P_{X|Y}(x|y) \geq \lambda_{\max} \text{ or } -\log P_{X|Y}(x|y) < \lambda_{\min}\}$$

Proof. We begin by analyzing the reliability of Protocol 1. Let $\rho(X, Y)$ denote the number of rounds after which the protocol stops when the observations are (X, Y) . An error occurs if $(X, Y) \in \mathcal{T}_0$ or if there exists a $\hat{x} \neq X$ such that $(\hat{x}, Y) \in \mathcal{T}_l$ and $F_{1j}(X) = F_{1j}(\hat{x})$ for all j such that $1 \leq j \leq l$, for some $l \leq \rho(X, Y)$. Note that for each $1 \leq j \leq L$,

$$|\{x : (x, y) \in \mathcal{T}_j\}| \leq \exp(\lambda_j + \Delta) \quad \forall y \in \mathcal{Y}.$$

Therefore, using a slight modification of the usual probability of error analysis for random binning, the probability of error for Protocol 1 is bounded above as

$$\begin{aligned} P_e &\leq P_{XY}(\mathcal{T}_0) + \sum_{x, y} P_{XY}(x, y) \sum_{l=1}^{\rho(x, y)} \sum_{\hat{x} \neq x} P(F_{1j}(x) = F_{1j}(\hat{x}), \forall 1 \leq j \leq l) \mathbb{1}((\hat{x}, y) \in \mathcal{T}_l) \\ &\leq P_{XY}(\mathcal{T}_0) + \sum_{x, y} P_{XY}(x, y) \sum_{l=1}^{\rho(x, y)} \sum_{\hat{x} \neq x} \frac{1}{M_1 \dots M_l} \mathbb{1}((\hat{x}, y) \in \mathcal{T}_l) \\ &\leq P_{XY}(\mathcal{T}_0) + \sum_{x, y} P_{XY}(x, y) \sum_{l=1}^{\rho(x, y)} 2^{-\lambda_l - \Delta - \gamma} |\{\hat{x} : (\hat{x}, y) \in \mathcal{T}_l\}| \\ &\leq P_{XY}(\mathcal{T}_0) + L2^{-\gamma}, \end{aligned} \tag{10}$$

where we have used the fact that $\log M_1 \dots M_l = \lambda_1 + l\Delta + \gamma = \lambda_l + \Delta + \gamma$.

We now establish the secrecy of the protocol. Our proof entails establishing secrecy of the protocol

conditioned on each realization $J = j$ of an appropriately defined RV, which roughly corresponds to the slice index for (X, Y) . Specifically, denote by \mathcal{E}_1 the set of (x, y) for which an error occurs in information reconciliation and by \mathcal{E}_2 the set

$$\mathcal{E}_2 := \{(x, y, z) : i_{XY}(x, y) - i_{XZ}(x, z) \leq \lambda + \Delta\},$$

which is the same as

$$\left\{ (x, y, z) : \log \frac{1}{P_{X|Z}(x|z)} - \log \frac{1}{P_{X|Y}(x|y)} \leq \lambda + \Delta \right\}$$

Let RV J taking values in the set $\{0, 1, \dots, L\}$ be defined as follows:

$$J = \begin{cases} 0, & \text{if } (X, Y) \in \mathcal{T}_0 \cup \mathcal{E}_1 \text{ or } (X, Y, Z) \in \mathcal{E}_2, \\ j & \text{if } (X, Y) \in \mathcal{T}_j \cap \mathcal{E}_1^c \text{ and } (X, Y, Z) \in \mathcal{E}_2^c, 1 \leq j \leq L. \end{cases}$$

While we have used random coding in the information reconciliation stage, it is only for the ease of proof and the encoder can be easily derandomized.⁸ For the remainder of the proof, we assume a deterministic encoder; in particular, J is a function of (X, Y, Z) .

We divide the indices $0 \leq j \leq L$ into good indices \mathcal{I}_g and the bad indices $\mathcal{I}_b = \mathcal{I}_g^c$, where

$$\mathcal{I}_g = \left\{ j : j > 0 \text{ and } P_J(j) \geq \frac{1}{L^2} \right\}.$$

Denoting by \mathbf{F}^l the communication up to l round of the protocol, *i.e.*, $\mathbf{F}^l := \{(F_{1j}, F_{2j}), 1 \leq j \leq l\}$ and by $\mathbf{F} = \mathbf{F}^{\rho(X, Y)}$ the overall communication, we have

$$\begin{aligned} & \|P_{K\mathbf{F}ZS} - P_{\text{unif}}P_{\mathbf{F}ZS}\|_1 \\ & \leq \|P_{K\mathbf{F}ZSJ} - P_{\text{unif}}P_{\mathbf{F}ZSJ}\|_1 \\ & \leq P(J \in \mathcal{I}_b) + \sum_{j \in \mathcal{I}_g} P_J(j) \|P_{K\mathbf{F}ZS|J=j} - P_{\text{unif}}P_{\mathbf{F}ZS|J=j}\|_1 \\ & \leq P_{XY}(\mathcal{T}_0 \cup \mathcal{E}_1) + P_{XYZ}(\mathcal{E}_2) + \frac{1}{L} + \sum_{j \in \mathcal{I}_g} P_J(j) \|P_{K\mathbf{F}ZS|J=j} - P_{\text{unif}}P_{\mathbf{F}ZS|J=j}\|_1. \end{aligned} \quad (11)$$

To bound each term $\|P_{K\mathbf{F}ZS|J=j} - P_{\text{unif}}P_{\mathbf{F}ZS|J=j}\|_1, j \in \mathcal{I}_g$, first note that under each event $J = j \in \mathcal{I}_g$ information reconciliation succeeds and $\mathbf{F} = \mathbf{F}^j$. Furthermore, the number of possible transcripts sent in the information reconciliation stage up to j th round, *i.e.*, the cardinality $\|\mathbf{F}^j\|$ of the range of RV \mathbf{F}^j ,

⁸Since the final error probability is given by the expected probability of error, where the expectation is over the source distribution and the additional shared randomness used in communication, there exists a realization of the shared randomness for which the same expected error performance with respect to the source distribution is attained.

satisfies (cf. Remark 3)

$$\log \|\mathbf{F}^j\| \leq \lambda_j + \Delta + \gamma + \log j.$$

Let P_j be the probability distribution of X, Y, Z given $J = j$, *i.e.*,

$$P_j(x, y, z) := \frac{P_{XYZ}(x, y, z) \mathbf{1}(J(x, y, z) = j)}{P_J(j)}, \quad x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}, 0 \leq j \leq L.$$

With $P_{j,XZ}$ denoting the marginal on $\mathcal{X} \times \mathcal{Z}$ induced by P_j , for all $j \in \mathcal{I}_g$, we have

$$\begin{aligned} \log \frac{P_{j,XZ}(x, z)}{P_Z(z)} &= \log \frac{\sum_y P_{XYZ}(x, y, z) \mathbf{1}(J(x, y, z) = j)}{P_J(j) P_Z(z)} \\ &\leq \log \frac{\sum_y 2^{-\lambda-\Delta} P_{X|Y}(x|y) P_{Y|XZ}(y|x, z) \mathbf{1}(J(x, y, z) = j)}{P_J(j)} \\ &\leq \log \frac{\sum_y 2^{-\lambda_i-\lambda-\Delta} P_{Y|XZ}(y|x, z) \mathbf{1}(J(x, y, z) = j)}{P_J(j)} \\ &\leq \log \frac{2^{-\lambda_i-\lambda-\Delta}}{P_J(j)} \\ &\leq -\lambda_i - \lambda - \Delta + 2 \log L, \end{aligned}$$

where the first inequality holds since $J(x, y, z) > 0$ implies $(x, y, z) \in \mathcal{E}_2^c$, the second inequality holds since $J(x, y, z) = j$ implies $(x, y) \in \mathcal{T}_j$, and the last inequality holds since $j \in \mathcal{I}_g$ implies $P_J(j) > \frac{1}{L^2}$. Thus, we obtain the following bound on $H_{\min}(P_{j,XZ}|P_Z)$:

$$H_{\min}(P_{j,XZ}|P_Z) \geq \lambda_j + \lambda + \Delta - 2 \log L.$$

Therefore, noting that S is independent of (X, Z, \mathbf{F}, J) and using Lemma 4, we get

$$\begin{aligned} \left\| P_{K\mathbf{F}ZS|J=j} - P_{\text{unif}} P_{\mathbf{F}ZS|J=j} \right\|_1 &= \left\| P_{K\mathbf{F}^jZS|J=j} - P_{\text{unif}} P_{\mathbf{F}^jZS|J=j} \right\|_1 \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| \|\mathbf{F}^j\| 2^{-H_{\min}(P_{j,XZ}|P_Z)}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda-\gamma-3 \log L)}}, \quad j \in \mathcal{I}_g. \end{aligned}$$

which gives the claimed secrecy by using the definition of \mathcal{E}_2 and bounding $P_{XY}(\mathcal{T}_0 \cup \mathcal{E}_1)$ using the union bound as in (10). \square

Thus, when the secret key length $\log |\mathcal{K}| \approx \lambda$, the reliability parameter ϵ for Protocol 1 can be made very small by appropriately choosing parameters λ_{\min} and λ_{\max} , and the secrecy parameter δ can be made roughly as small as the tail-probability $P(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda)$. In fact, Proposition 2 allows us to shift this constraint on δ to a constraint on $\epsilon + \delta$ and Protocol 1 yields an (ϵ, δ) -SK of length

roughly equal to λ as long as $\epsilon + \delta$ is greater than $\mathbb{P}(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda)$. Formally, we have the following simple corollary of Theorem 5.

Corollary 6. For $\lambda_{\max}, \lambda_{\min}, \Delta > 0$ with $\lambda_{\max} \geq \lambda_{\min}$, let

$$L = \frac{\lambda_{\max} - \lambda_{\min}}{\Delta},$$

and let

$$\mathcal{T}_0 := \{(x, y) : -\log P_{X|Y}(x|y) \geq \lambda_{\max} \text{ or } -\log P_{X|Y}(x|y) < \lambda_{\min}\}.$$

Then, for every $\lambda \geq 0$ and every ϵ and δ satisfying

$$\epsilon \geq \mathbb{P}_{XY}(\mathcal{T}_0) + \frac{1}{4} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}, \quad (12)$$

$$\epsilon + \delta \geq \mathbb{P}(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda + \Delta) + \frac{1}{L} + 2\mathbb{P}_{XY}(\mathcal{T}_0) + \frac{3}{2} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}, \quad (13)$$

there exists an (ϵ, δ) -SK K taking values in \mathcal{K} .

Proof. Let

$$\eta(\gamma) := \mathbb{P}_{XY}(\mathcal{T}_0) + L2^{-\gamma},$$

$$\alpha(\gamma) := \mathbb{P}(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda + \Delta) + \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - \gamma - 3 \log L)}} + \frac{1}{L} + \mathbb{P}_{XY}(\mathcal{T}_0) + L2^{-\gamma}.$$

We first optimize

$$\begin{aligned} & \eta(\gamma) + \alpha(\gamma) \\ &= \mathbb{P}(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda + \Delta) + \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - \gamma - 3 \log L)}} + \frac{1}{L} + 2\mathbb{P}_{XY}(\mathcal{T}_0) + 2L2^{-\gamma} \end{aligned}$$

over γ . By setting $a = L2^{-\gamma}$ and by noting that the function $f(a) = 2a + \frac{a^{-1/2}L^2}{2} \sqrt{|\mathcal{K}| 2^{-\lambda}}$ has minimum value $\frac{3}{2} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}$ with $a = \frac{1}{4} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}$, the minimum of $\eta(\gamma) + \alpha(\gamma)$ is achieved when

$$\eta = \eta^* := \mathbb{P}_{XY}(\mathcal{T}_0) + \frac{1}{4} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}},$$

$$\alpha = \alpha^* := \mathbb{P}(i_{XY}(X, Y) - i_{XZ}(X, Z) \leq \lambda + \Delta) + \frac{1}{L} + \mathbb{P}_{XY}(\mathcal{T}_0) + \frac{5}{4} \left(L^4 |\mathcal{K}| 2^{-\lambda} \right)^{\frac{1}{3}}.$$

The corollary follows by applying Proposition 2 with $\eta = \eta^*$ and α^* to the resulting (η^*, α^*) -SK. \square

V. SECRET KEY CAPACITY FOR GENERAL SOURCES

In this section we will establish the secret key capacity for a sequence of general sources (X_n, Y_n, Z_n) with joint distribution⁹ $P_{X_n Y_n Z_n}$. The secret key capacity for general sources is defined as follows [19], [1], [4].

Definition 3. The secret key capacity C is defined as

$$C := \sup_{\epsilon_n, \delta_n} \liminf_{n \rightarrow \infty} \frac{1}{n} S_{\epsilon_n, \delta_n} (X_n, Y_n | Z_n),$$

where the sup is over all $\epsilon_n, \delta_n \geq 0$ such that

$$\lim_{n \rightarrow \infty} \epsilon_n + \delta_n = 0.$$

To state our result, we need the following concepts from the *information spectrum* method; see [10] for a detailed account. For RVs $(X_n, Y_n, Z_n)_{n=1}^{\infty}$, the *inf-conditional entropy rate* $\underline{H}(\mathbf{X} | \mathbf{Y})$ and the *sup-conditional entropy rate* $\overline{H}(\mathbf{X} | \mathbf{Y})$ are defined as follows:

$$\begin{aligned} \underline{H}(\mathbf{X} | \mathbf{Y}) &= \sup \left\{ \alpha \mid \lim_{n \rightarrow \infty} \mathbb{P} \left(-\frac{1}{n} \log P_{X_n | Y_n} (X_n | Y_n) < \alpha \right) = 0 \right\}, \\ \overline{H}(\mathbf{X} | \mathbf{Y}) &= \inf \left\{ \alpha \mid \lim_{n \rightarrow \infty} \mathbb{P} \left(-\frac{1}{n} \log P_{X_n | Y_n} (X_n | Y_n) > \alpha \right) = 0 \right\}. \end{aligned}$$

Similarly, the *inf-conditional information rate* $\underline{I}(\mathbf{X} \wedge \mathbf{Y} | \mathbf{Z})$ is defined as

$$\underline{I}(\mathbf{X} \wedge \mathbf{Y} | \mathbf{Z}) = \sup \left\{ \alpha \mid \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} i(X_n, Y_n | Z_n) < \alpha \right) = 0 \right\},$$

where, with a slight abuse of notation, $i(X_n, Y_n | Z_n)$ denotes the conditional information density

$$i(X_n, Y_n | Z_n) = \log \frac{P_{X_n Y_n | Z_n} (X_n, Y_n | Z_n)}{P_{X_n | Z_n} (X_n | Z_n) P_{Y_n | Z_n} (Y_n | Z_n)}.$$

We also need the following result credited to Verdú.

Lemma 7. [10, Theorem 4.1.1] *For every ϵ_n such that*

$$\lim_{n \rightarrow \infty} \epsilon_n = 0,$$

it holds that

$$\liminf_n -\frac{1}{n} \log \beta_{\epsilon_n} (P_{X_n Y_n Z_n}, P_{X_n | Z_n} P_{Y_n | Z_n} P_{Z_n}) \leq \underline{I}(\mathbf{X} \wedge \mathbf{Y} | \mathbf{Z}),$$

⁹The distributions $P_{X_n Y_n Z_n}$ need not satisfy the consistency conditions.

where β_{ϵ_n} is defined in (8).

Our result below characterizes the secret key capacity C for general sources for the special case when $X_n \text{--}\ominus Y_n \text{--}\ominus Z_n$ is a Markov chain.

Theorem 8. *For a sequence of sources $\{X_n, Y_n, Z_n\}_{n=1}^{\infty}$ such that $X_n \text{--}\ominus Y_n \text{--}\ominus Z_n$ form a Markov chain for all n , the secret key capacity C is given by¹⁰*

$$C = \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}).$$

Proof. Applying Theorem 3 with $\eta = \eta_n = n^{-1}$, along with Lemma 7, gives

$$C \leq \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}).$$

For the other direction, we construct a sequence of (ϵ_n, δ_n) -SKs $K = K_n$ with $\epsilon_n, \delta_n \rightarrow 0$ and rate approximately $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$. Indeed, in Theorem 5 choose

$$\lambda_{\max} = n(\overline{H}(\mathbf{X} \mid \mathbf{Y}) + \Delta),$$

$$\lambda_{\min} = n(\underline{H}(\mathbf{X} \mid \mathbf{Y}) - \Delta),$$

$$\gamma = \gamma_n = n\Delta/2,$$

$$\lambda = \lambda_n = n(\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) - \Delta);$$

thus,

$$L = L_n = \frac{n(\overline{H}(\mathbf{X} \mid \mathbf{Y}) - \underline{H}(\mathbf{X} \mid \mathbf{Y}) + 2\Delta)}{\Delta}.$$

Since $i_{XY}(X, Y) - i_{XZ}(X, Z) = i(X, Y \mid Z)$ if $X \text{--}\ominus Y \text{--}\ominus Z$ form a Markov chain, there exists an (ϵ_n, δ_n) -SK K_n of rate given by

$$\begin{aligned} \frac{1}{n} \log |\mathcal{K}| &= \frac{1}{n} (\lambda_n - 3 \log L_n) - \Delta \\ &= \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) - 2\Delta - o(n), \end{aligned}$$

such that $\epsilon_n, \delta_n \rightarrow 0$ as $n \rightarrow \infty$. Rates arbitrarily close to $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$ are achieved by this scheme as $\Delta > 0$ is arbitrary. \square

In the achievability part of the proof above, we actually show that, in general, our protocol generates

¹⁰We assume that $\overline{H}(\mathbf{X} \mid \mathbf{Y}) < \infty$.

a secret key of rate

$$\sup \left\{ \alpha \mid \lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{1}{n} [i(X_n, Y_n) - i(X_n, Z_n)] < \alpha \right) = 0 \right\},$$

which matches the converse bound of $\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z})$ in the special case when $X_n \ominus Y_n \ominus Z_n$ holds.

VI. SECOND-ORDER ASYMPTOTICS OF SECRET KEY RATES

The results of the previous section show that for with $\epsilon_n, \delta_n \rightarrow 0$, the largest length $S_{\epsilon_n, \delta_n}(X_n, Y_n \mid Z_n)$ of an (ϵ_n, δ_n) -SK K is

$$\sup_{\epsilon_n, \delta_n} S_{\epsilon_n, \delta_n}(X_n, Y_n \mid Z_n) = n \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) + o(n), \quad (14)$$

if $X_n \ominus Y_n \ominus Z_n$ form a Markov chain. For the case when $(X_n, Y_n, Z_n) = (X^n, Y^n, Z^n)$ is the n -IID repetition of (X, Y, Z) where $X \ominus Y \ominus Z$, we have

$$\underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) = I(X \wedge Y \mid Z).$$

Furthermore, (14) holds even without $\epsilon_n, \delta_n \rightarrow 0$. In fact, a finer asymptotic analysis is possible and the second-order asymptotic term in the maximum length of an (ϵ, δ) -SK can be established; this is the subject-matter of the current section.

Let

$$V := \mathbb{V}\text{ar} [i(X, Y \mid Z)],$$

and let

$$Q(a) := \int_a^\infty \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{t^2}{2} \right] dt$$

be the tail probability of the standard Gaussian distribution. Under the assumptions

$$V_{X|Y} := \mathbb{V}\text{ar}[-\log P_{X|Y}(X|Y)] < \infty, \quad (15)$$

$$T := \mathbb{E} \left[|i(X, Y \mid Y) - I(X \wedge Y \mid Z)|^3 \right] < \infty, \quad (16)$$

the result below establishes the second-order asymptotic term in $S_{\epsilon, \delta}(X^n, Y^n \mid Z^n)$.

Theorem 9. *For every $\epsilon, \delta > 0$ such that $\epsilon + \delta < 1$ and IID RVs (X^n, Y^n, Z^n) such that $X \ominus Y \ominus Z$ is a Markov chain, we have*

$$S_{\epsilon, \delta}(X^n, Y^n \mid Z^n) = nI(X \wedge Y \mid Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta) \pm \mathcal{O}(\log n),$$

Proof. For the converse part, we proceed along the lines of [23, Lemma 58]. Recall the following simple bound for $\beta_\epsilon(\mathsf{P}, \mathsf{Q})$ (cf. [10, Lemma 4.1.2]):

$$-\log \beta_\epsilon(\mathsf{P}, \mathsf{Q}) \leq \lambda - \log \left(\mathsf{P} \left(\left\{ x : \log \frac{\mathsf{P}(x)}{\mathsf{Q}(x)} \leq \lambda \right\} \right) - \epsilon \right).$$

Thus, applying Theorem 3 with $\mathsf{P}_{XYZ} = \mathsf{P}_{X^n Y^n Z^n}$, $\mathsf{Q}_{XYZ} = \mathsf{P}_{X^n | Z^n} \mathsf{P}_{Y^n Z^n}$, and $\eta = \eta_n = n^{-1/2}$, and choosing

$$\lambda = nI(X \wedge Y | Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta + \theta_n),$$

where

$$\theta_n = \frac{2}{\sqrt{n}} + \frac{T^3}{2V^{3/2}\sqrt{n}},$$

we get by the Berry-Esséen theorem (cf. [7], [26]) that

$$\mathsf{P}(i(X^n, Y^n) - i(X^n, Z^n) \leq \lambda) \geq \epsilon + \delta + \frac{2}{\sqrt{n}},$$

which implies

$$\begin{aligned} S_{\epsilon, \delta}(X^n, Y^n | Z^n) &\leq \lambda - \log \left(\mathsf{P}(i(X^n, Y^n) - i(X^n, Z^n) \leq \lambda) - \epsilon - \delta - n^{-1/2} \right) + \log n \\ &\leq nI(X \wedge Y | Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta + \theta_n) + \frac{3}{2} \log n. \end{aligned} \quad (17)$$

Thus, we have the desired converse by using Taylor approximation of $Q(\cdot)$ to remove θ_n .

For the direct part, we use Corollary 6 by setting

$$\lambda_{\max} = n(H(X|Y) + \Delta/2),$$

$$\lambda_{\min} = n(H(X|Y) - \Delta/2),$$

$$\lambda = nI(X \wedge Y | Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta - \theta'_n) - \Delta,$$

and

$$\log |\mathcal{K}| = nI(X \wedge Y | Z) - \sqrt{nV}Q^{-1}(\epsilon + \delta - \theta'_n) - \frac{11}{2} \log n - \Delta, \quad (18)$$

where $0 < \Delta < 2H(X | Y)$, and

$$\theta'_n = \frac{8V_{X|Y}}{n\Delta^2} + \frac{T^3}{2V^{3/2}\sqrt{n}} + \frac{1}{n} + \frac{3}{2\sqrt{n}}.$$

Note that $L = n$. Upon bounding the term $\mathsf{P}_{XY}(\mathcal{T}_0)$ in (12) and (13) by $\frac{4V_{X|Y}}{n\Delta^2}$ using Chebyshev's inequality, the condition (12) is satisfied for sufficiently large n . Furthermore, upon bounding the first

term of (13) by the Berry-Esséen theorem, the condition (13) is also satisfied. Thus, it follows that there exists an (ϵ, δ) -SK K taking values on \mathcal{K} . The direct part follows by using Taylor approximation of $Q(\cdot)$ to remove θ'_n . \square

Remark 4. Note that a standard noninteractive secret key agreement protocol based on information reconciliation and privacy amplification (cf. [25]) only gives the following suboptimal achievability bound on the second-order asymptotic term:

$$S_{\epsilon, \delta}(X^n, Y^n | Z^n) \geq nI(X \wedge Y | Z) - \sqrt{nV_{X|Y}}Q^{-1}(\epsilon) - \sqrt{nV_{X|Z}}Q^{-1}(\delta) + o(\sqrt{n}),$$

where $V_{X|Y}$ and $V_{X|Z}$ are the variances of the conditional log-likelihoods of X given Y and Z respectively (cf. (15)).

We close this section with a numerical example that illustrates the utility of our bounds in characterizing the gap to secret key capacity at a fixed n .

Example 1 (Gap to secret key capacity). For $\alpha_0, \alpha_1 \in (0, 1/2)$, let B_0 and B_1 be independent random bits taking value 1 with probability α_0 and α_1 , respectively. Consider binary X, Y, Z where Z is a uniform random bit independent jointly of B_0 and B_1 , $Y = Z \oplus B_0$, and $X = Y \oplus B_1$. We consider the rate $S_{\epsilon, \delta}(X^n, Y^n | Z^n)/n$ of an (ϵ, δ) -SK that can be generated using n IID copies of X and Y when the eavesdropper observes Z^n . The following quantities, needed to evaluate (17) and (18), can be easily evaluated:

$$I(X \wedge Y | Z) = h(\alpha_0 * \alpha_1) - h(\alpha_1), \quad V = \mu_2, \quad T = \mu_3,$$

$$V_{X|Y} = \alpha_1(\log \alpha_1 - h(\alpha_1))^2 + (1 - \alpha_1)(\log(1 - \alpha_1) - h(\alpha_1))^2,$$

where μ_r is the r th central moment of $i(X, Y | Z)$ and is given by

$$\begin{aligned} \mu_r = & \alpha_0 \alpha_1 \left| \log \frac{\alpha_1}{1 - \alpha_0 * \alpha_1} - I(X \wedge Y | Z) \right|^r + (1 - \alpha_0)(1 - \alpha_1) \left| \log \frac{1 - \alpha_1}{1 - \alpha_0 * \alpha_1} - I(X \wedge Y | Z) \right|^r \\ & + (1 - \alpha_0)\alpha_1 \left| \log \frac{\alpha_1}{\alpha_0 * \alpha_1} - I(X \wedge Y | Z) \right|^r + \alpha_0(1 - \alpha_1) \left| \log \frac{1 - \alpha_1}{\alpha_0 * \alpha_1} - I(X \wedge Y | Z) \right|^r, \end{aligned}$$

$h(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy function and $\alpha_0 * \alpha_1 = \alpha_0(1 - \alpha_1) + (1 - \alpha_0)\alpha_1$. In Figure 1 (given in Section I), we plot the upper bound on $S_{\epsilon, \delta}(X^n, Y^n | Z^n)/n$ resulting from (17) and the lower bound resulting from (17) with $\Delta = 1$ for $\alpha_0 = 0.25$ and $\alpha_1 = 0.125$.

VII. DISCUSSION: IS INTERACTION NECESSARY?

In contrast to the protocols in [19], [1], [4], [25], our proposed Protocol 1 for secret key agreement is interactive. In fact, the protocol requires as many rounds of interaction as the number of slices L , which can be pretty large in general. For instance, to obtain the second-order asymptotic term in the previous section, we chose $L = n$. In Appendix C, we present an alternative protocol which requires only 1-bit of feedback and, in the special case when Z is constant, achieves the asymptotic results of Sections V and VI. But is interaction necessary for attaining our asymptotic results? Below we present an example where none of the known (noninteractive) secret key agreement protocols achieves the general capacity of Theorem 8, suggesting that perhaps interaction is necessary.

For $i = 1, 2$, let (X_i^n, Y_i^n, Z_i^n) be IID with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0, 1\}$ such that

$$P_{X_i^n Y_i^n Z_i^n}(x^n, y^n, z^n) = \frac{1}{2^n} W_i^n(y^n|x^n) V^n(z^n|y^n),$$

where W_i and V , respectively, are binary symmetric channels with crossover probabilities p_i and q . Let (X_n, Y_n, Z_n) be the mixed source given by

$$\begin{aligned} P_{X_n Y_n Z_n}(x^n, y^n, z^n) &= \frac{1}{2} P_{X_1^n Y_1^n Z_1^n}(x^n, y^n, z^n) + \frac{1}{2} P_{X_2^n Y_2^n Z_2^n}(x^n, y^n, z^n) \\ &= \frac{1}{2^n} \left[\frac{1}{2} W_1^n(y^n|x^n) + \frac{1}{2} W_2^n(y^n|x^n) \right] V^n(z^n|y^n). \end{aligned}$$

Note that $X_n \ominus Y_n \ominus Z_n$ forms a Markov chain. Suppose that $0 < p_1 < p_2 < \frac{1}{2}$. Then, we have

$$\begin{aligned} \underline{I}(\mathbf{X} \wedge \mathbf{Y} \mid \mathbf{Z}) &= \min[H(X_1|Z_1) - H(X_1|Y_1), H(X_2|Z_2) - H(X_2|Y_2)] \\ &= \min[h(p_1 * q) - h(p_1), h(p_2 * q) - h(p_2)] \\ &= h(p_2 * q) - h(p_2), \end{aligned}$$

where $h(\cdot)$ is the binary entropy function and $*$ is binary convolution. Using a standard noninteractive secret key agreement protocol based on information reconciliation and privacy amplification (cf. [25]), we can achieve only

$$\begin{aligned} \underline{H}(\mathbf{X} \mid \mathbf{Z}) - \overline{H}(\mathbf{X} \mid \mathbf{Y}) &= \min[H(X_1|Z_1), H(X_2|Z_2)] - \max[H(X_1|Y_1), H(X_2|Y_2)] \\ &= H(X_1|Z_1) - H(X_2|Y_2) \\ &= h(p_1 * q) - h(p_2), \end{aligned}$$

which is less than the general secret key capacity of Theorem 8. Proving a precise limitation result for noninteractive protocols is a direction for future research.

APPENDIX

A. Proof of Theorem 3

The definition of a secret key used in [33], [32] is different from the one in Definition 1, and it conveniently combines the secrecy and the reliability requirements into a single expression. Instead of considering a separate RV K , the alternative definition directly works with the estimates K_x and K_y . Specifically, let K_x and K_y be functions of (U_x, X, \mathbf{F}) and (U_y, Y, \mathbf{F}) , respectively, where \mathbf{F} is an interactive communication. Then, RVs K_x and K_y with a common range \mathcal{K} constitute an ϵ -secret key (ϵ -SK) if

$$\left\| \mathbb{P}_{K_x K_y \mathbf{F} Z} - \mathbb{P}_{\text{unif}}^{(2)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1 \leq \epsilon, \quad (19)$$

where, for a pmf \mathbb{P} on \mathcal{X} , $\mathbb{P}^{(m)}$ denotes its extension to \mathcal{X}^m given by

$$\mathbb{P}^{(m)}(x_1, \dots, x_m) = \mathbb{P}(x) \mathbb{1}(x_1 = \dots = x_m), \quad (x_1, \dots, x_m) \in \mathcal{X}^m.$$

Note that the alternative definition captures reliability condition $\mathbb{P}(K_x = K_y) \geq 1 - \epsilon$ by requiring that the joint distribution $\mathbb{P}_{K_x K_y}$ is close to a uniform distribution on the diagonal of $\mathcal{K} \times \mathcal{K}$. The upper bound in [33], [32] holds under this alternative definition of a secret key. However, the next lemma says that this alternative definition is closely related to our Definition 1.

Lemma 10. *Given $\epsilon, \delta \in [0, 1)$ and an (ϵ, δ) -SK K , the local estimates K_x and K_y satisfy (19) with $\epsilon + \delta$, i.e.,*

$$\left\| \mathbb{P}_{K_x K_y \mathbf{F} Z} - \mathbb{P}_{\text{unif}}^{(2)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1 \leq \epsilon + \delta.$$

Conversely, if K_x and K_y satisfy (19), either K_x or K_y constitutes an (ϵ, ϵ) -SK.

Proof. We prove the direct part first. For an (ϵ, δ) -SK K ,

$$\begin{aligned} & \left\| \mathbb{P}_{K_x K_y \mathbf{F} Z} - \mathbb{P}_{\text{unif}}^{(2)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1 \\ & \leq \left\| \mathbb{P}_{K K_x K_y \mathbf{F} Z} - \mathbb{P}_{\text{unif}}^{(3)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1 \\ & \leq \left\| \mathbb{P}_{K K_x K_y \mathbf{F} Z} - \mathbb{P}_{K|\mathbf{F} Z}^{(3)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1 + \left\| \mathbb{P}_{K|\mathbf{F} Z}^{(3)} \times \mathbb{P}_{\mathbf{F} Z} - \mathbb{P}_{\text{unif}}^{(3)} \times \mathbb{P}_{\mathbf{F} Z} \right\|_1. \end{aligned}$$

Since

$$\|P - Q\|_1 = Q(\{x : Q(x) \geq P(x)\}) - P(\{x : Q(x) \geq P(x)\})$$

and

$$\begin{aligned} & \{(k, k_x, k_y, f, z) : P_{K|\mathbf{FZ}}^{(3)}(k, k_x, k_y|f, z) \geq P_{KK_xK_y|\mathbf{FZ}}(k, k_x, k_y|f, z)\} \\ &= \{(k, k_x, k_y, \mathbf{f}, z) : k = k_x = k_y\}, \end{aligned}$$

the first term on the right-side above satisfies

$$\left\| P_{KK_xK_y\mathbf{FZ}} - P_{K|\mathbf{FZ}}^{(3)} \times P_{\mathbf{FZ}} \right\|_1 = 1 - P(K = K_x = K_y) \leq \epsilon. \quad (20)$$

Furthermore, the second term satisfies

$$\begin{aligned} & \left\| P_{K|\mathbf{FZ}}^{(3)} \times P_{\mathbf{FZ}} - P_{\text{unif}}^{(3)} \times P_{\mathbf{FZ}} \right\|_1 \\ &= \sum_{k, k_x, k_y, f, z} P_{\mathbf{FZ}}(f, z) \left| P_{K|\mathbf{FZ}}(k|f, z) \mathbb{1}(k = k_x = k_y) - \mathbb{1}(k = k_x = k_y) \frac{1}{|\mathcal{K}|} \right| \\ &= \left\| P_{K|\mathbf{FZ}} \times P_{\mathbf{FZ}} - P_{\text{unif}} \times P_{\mathbf{FZ}} \right\|_1 \\ &\leq \delta, \end{aligned}$$

where the last inequality is by the δ -secrecy condition K . Combining the bounds on the two terms above, the direct part follows.

For the converse, ϵ -secrecy of K_x (or K_y) holds since by the monotonicity of the variational distance

$$\|P_{K_x\mathbf{FZ}} - P_{\text{unif}} \times P_{\mathbf{FZ}}\|_1 \leq \left\| P_{K_xK_y\mathbf{FZ}} - P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}} \right\|_1 \leq \epsilon.$$

The ϵ -reliability condition, too, follows from the triangle inequality upon observing that

$$\begin{aligned} \left\| P_{K_xK_y} - P_{\text{unif}}^{(2)} \right\|_1 &= \sum_{k_x, k_y} \left| P_{K_xK_y}(k_x, k_y) - \mathbb{1}(k_x = k_y) \frac{1}{|\mathcal{K}|} \right| \\ &\geq \sum_{k_x \neq k_y} P_{K_xK_y}(k_x, k_y) \\ &= P(K_x \neq K_y). \end{aligned}$$

□

To prove Theorem 3, we first relate the length of a secret key satisfying (19) to the exponent of the probability of error of type II in a binary hypothesis testing problem where an observer of $(K_x, K_y, \mathbf{F}, Z)$

seeks to find out if the underlying distribution was P_{XYZ} of $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_Z$. This result is stated next.

Lemma 11. *For an ϵ -SK (K_x, K_y) satisfying (19) generated by an interactive communication \mathbf{F} , let $W_{K_x K_y \mathbf{F} | XYZ}$ denote the resulting conditional distribution on (K_x, K_y, \mathbf{F}) given (X, Y, Z) . Then, for every $0 < \eta < 1 - \epsilon$ and every $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_Z$, we have*

$$\log |\mathcal{K}| \leq -\log \beta_{\epsilon+\eta}(P_{K_x K_y \mathbf{F} Z}, Q_{K_x K_y \mathbf{F} Z}) + 2 \log(1/\eta), \quad (21)$$

where $Q_{K_x K_y \mathbf{F} Z}$ is the marginal of $(K_x, K_y, \mathbf{F}, Z)$ of the joint distribution

$$Q_{K_x K_y \mathbf{F} XYZ} = Q_{XYZ} W_{K_x K_y \mathbf{F} | XYZ}.$$

To prove Lemma 11, we need the following basic property of interactive communication (cf. [31]).

Lemma 12 (Interactive communication property). *Given $Q_{XYZ} = Q_{X|Z}Q_{Y|Z}Q_Z$ and an interactive communication \mathbf{F} , the following holds:*

$$Q_{XY|\mathbf{F}Z} = Q_{X|\mathbf{F}Z} \times Q_{Y|\mathbf{F}Z},$$

i.e., conditionally independent observations remain so when conditioned additionally on an interactive communication.

Proof of Lemma 11: We establish (21) by constructing a test for the hypothesis testing problem with null hypothesis $P = P_{K_x K_y \mathbf{F} Z}$ and alternative hypothesis $Q = Q_{K_x K_y \mathbf{F} Z}$. Specifically, we use a deterministic test with the following acceptance region (for the null hypothesis)¹¹:

$$\mathcal{A} := \left\{ (k_x, k_y, f, z) : \log \frac{P_{\text{unif}}^{(2)}(k_x, k_y)}{Q_{K_x K_y \mathbf{F} Z}(k_x, k_y | f, z)} \geq \lambda \right\},$$

where

$$\lambda = \log |\mathcal{K}| - 2 \log(1/\eta).$$

¹¹The values (k_x, k_y, f, z) with $Q_{K_x K_y \mathbf{F} Z}(k_x, k_y | f, z) = 0$ are included in \mathcal{A} .

For this test, the probability of type II is bounded above as

$$\begin{aligned}
Q_{K_x K_y \mathbf{FZ}}(\mathcal{A}) &= \sum_{f,z} Q_{\mathbf{FZ}}(f,z) \sum_{\substack{k_x, k_y: \\ (k_x, k_y, f, z) \in \mathcal{A}}} Q_{K_x K_y | \mathbf{FZ}}(k_x, k_y | f, z) \\
&\leq 2^{-\lambda} \sum_{f,z} Q_{\mathbf{FZ}}(f,z) \sum_{k_x, k_y} P_{\text{unif}}^{(2)}(k_x, k_y) \\
&= \frac{1}{|\mathcal{K}| \eta^2}.
\end{aligned} \tag{22}$$

On the other hand, the probability of error of type I is bounded above as

$$\begin{aligned}
P_{K_x K_y \mathbf{FZ}}(\mathcal{A}^c) &\leq \left\| P_{K_x K_y \mathbf{FZ}} - P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}} \right\|_1 + P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}}(\mathcal{A}^c) \\
&\leq \epsilon + P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}}(\mathcal{A}^c),
\end{aligned} \tag{23}$$

where the first inequality follows from the definition of variational distance, and the second is a consequence of the security criterion (19). The second term above can be expressed as follows:

$$\begin{aligned}
P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}}(\mathcal{A}^c) &= \sum_{f,z} P_{\mathbf{FZ}}(f,z) \frac{1}{|\mathcal{K}|} \sum_k \mathbf{1}((k, k, f, z) \in \mathcal{A}^c) \\
&= \sum_{f,z} P_{\mathbf{FZ}}(f,z) \frac{1}{|\mathcal{K}|} \sum_k \mathbf{1}(Q_{K_x K_y | \mathbf{FZ}}(k, k, |f, z) |\mathcal{K}|^2 \eta^2 > 1).
\end{aligned} \tag{24}$$

The inner sum can be further upper bounded as

$$\begin{aligned}
\sum_k \mathbf{1}(Q_{K_x K_y | \mathbf{FZ}}(k, k, |f, z) |\mathcal{K}|^2 \eta^2 > 1) &\leq \sum_k (Q_{K_x K_y | \mathbf{FZ}}(k, k, |f, z) |\mathcal{K}|^2 \eta^2)^{\frac{1}{2}} \\
&= |\mathcal{K}| \eta \sum_k Q_{K_x K_y | \mathbf{FZ}}(k, k, |f, z)^{\frac{1}{2}} \\
&= |\mathcal{K}| \eta \sum_k Q_{K_x | \mathbf{FZ}}(k | f, z)^{\frac{1}{2}} Q_{K_y | \mathbf{FZ}}(k | f, z)^{\frac{1}{2}},
\end{aligned} \tag{25}$$

where the previous equality uses Lemma 12 and the fact that given \mathbf{F} , K_x and K_y are functions of (X, U_x) and (Y, U_y) , respectively. Next, an application of the Cauchy-Schwartz inequality to the sum on the right-side of (25) yields

$$\begin{aligned}
\sum_k Q_{K_x | \mathbf{FZ}}(k | f, z)^{\frac{1}{2}} Q_{K_y | \mathbf{FZ}}(k | f, z)^{\frac{1}{2}} &\leq \left(\sum_{k_x} Q_{K_x | \mathbf{FZ}}(k_x | f, z) \right)^{\frac{1}{2}} \left(\sum_{k_y} Q_{K_y | \mathbf{FZ}}(k_y | f, z) \right)^{\frac{1}{2}} \\
&= 1.
\end{aligned} \tag{26}$$

Upon combining (24)-(26), we obtain

$$P_{\text{unif}}^{(2)} \times P_{\mathbf{FZ}}(\mathcal{A}^c) \leq \eta,$$

which along with (23) gives

$$P_{K_x K_y \mathbf{FZ}}(\mathcal{A}^c) \leq \epsilon + \eta. \quad (27)$$

It follows from (27) and (22) that

$$\beta_{\epsilon+\eta}(P_{K_x K_y \mathbf{FZ}}, Q_{K_x K_y \mathbf{FZ}}) \leq \frac{1}{|\mathcal{K}|\eta^2},$$

which completes the proof. \square

Finally, we derive the upper bound for $S_{\epsilon, \delta}(X, Y | Z)$ using the data processing property of β_ϵ : let W be a stochastic mapping from \mathcal{V} to \mathcal{V}' , i.e., for each $v \in \mathcal{V}$, $W(\cdot|v)$ is a distribution on \mathcal{V}' . Then, since the map W followed by a test on \mathcal{V}' can be regarded as a stochastic test on \mathcal{V} ,

$$\beta_\epsilon(P, Q) \leq \beta_\epsilon(P \circ W, Q \circ W), \quad (28)$$

where $(P \circ W)(v') = \sum_v P(v) W(v'|v)$.

Proof of Theorem 3: Using the data processing inequality (28) with $P = P_{XYZ}$, $Q = Q_{XYZ}$, and $W = W_{K_x K_y \mathbf{F}|XYZ}$, Lemma 11 implies that any (K_x, K_y) satisfying the secrecy criterion (19) must satisfy

$$\log |\mathcal{K}| \leq -\log \beta_{\epsilon+\eta}(P_{XYZ}, Q_{XYZ}) + 2 \log(1/\eta). \quad (29)$$

Furthermore, from Lemma 10, (ϵ, δ) -SK implies existence of local estimates K_x and K_y satisfying (19) with $(\epsilon + \delta)$ in place of ϵ . Thus, an (ϵ, δ) -SK with range \mathcal{K} must satisfy (29) with ϵ replaced by $(\epsilon + \delta)$, which completes the proof. \square

B. Proof of Lemma 4

Let $K_s = f_s(X)$ be the key for a fixed seed. By using the Cauchy-Schwarz inequality,

$$\begin{aligned} \|P_{K_s V Z} - P_{\text{unif}} P_{V Z}\|_1 &= \frac{1}{2} \sum_{k,v,z} \left| P_{K_s V Z}(k, v, z) - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z) \right| \\ &= \frac{1}{2} \sum_{k,v,z} \sqrt{Q_Z(z)} \left| \frac{P_{K_s V Z}(k, v, z) - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z)}{\sqrt{Q_Z(z)}} \right| \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| |\mathcal{V}| \sum_{k,v,z} \frac{\left(P_{K_s V Z}(k, v, z) - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z) \right)^2}{Q_Z(z)}}. \end{aligned}$$

Thus, by the concavity of $\sqrt{\cdot}$,

$$\|P_{K V Z S} - P_{\text{unif}} P_{V Z} P_S\|_1 \leq \frac{1}{2} \sqrt{|\mathcal{K}| |\mathcal{V}| \sum_{k,v,z,s} P_S(s) \frac{\left(P_{K_s V Z}(k, v, z) - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z) \right)^2}{Q_Z(z)}}.$$

The numerator of the sum can be rewritten as

$$\begin{aligned} &\sum_{k,s} P_S(s) \left(P_{K_s V Z}(k, v, z) - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z) \right)^2 \\ &= \sum_s P_S(s) \sum_k \left[P_{K_s V Z}(k, v, z)^2 - 2 P_{K_s V Z}(k, v, z) \frac{1}{|\mathcal{K}|} P_{V Z}(v, z) + \frac{1}{|\mathcal{K}|^2} P_{V Z}(v, z)^2 \right] \\ &= \sum_s P_S(s) \left[\sum_k P_{K_s V Z}(k, v, z)^2 - \frac{1}{|\mathcal{K}|} P_{V Z}(v, z)^2 \right] \\ &= \sum_s P_S(s) \left[\sum_{x,x'} P_{X V Z}(x, v, z) P_{X V Z}(x', v, z) \left\{ \mathbf{1}(f_s(x) = f_s(x')) - \frac{1}{|\mathcal{K}|} \right\} \right] \\ &= \sum_x P_{X V Z}(x, v, z)^2 \sum_s P_S(s) \left\{ 1 - \frac{1}{|\mathcal{K}|} \right\} \\ &\quad + \sum_{x \neq x'} P_{X V Z}(x, v, z) P_{X V Z}(x', v, z) \sum_s P_S(s) \left\{ \mathbf{1}(f_s(x) = f_s(x')) - \frac{1}{|\mathcal{K}|} \right\} \\ &\leq \sum_x P_{X V Z}(x, v, z)^2, \end{aligned}$$

where we used the property of two-universality (9) in the last inequality. Thus, we have

$$\begin{aligned}
\|P_{KVSZ} - P_{\text{unif}}P_{VZ}P_S\|_1 &\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,v,z} \frac{P_{XVZ}(x,v,z)^2}{Q_Z(z)}} \\
&\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,v,z} \frac{P_{XVZ}(x,v,z) P_{XZ}(x,z)}{Q_Z(z)}} \\
&= \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| \sum_{x,z} \frac{P_{XZ}(x,z)^2}{Q_Z(z)}} \\
&\leq \frac{1}{2} \sqrt{|\mathcal{K}||\mathcal{V}| 2^{-H_{\min}(P_{XZ}|Q_Z)}}.
\end{aligned}$$

□

Note that the last step in the proof above shows that it is, in fact, the conditional Rényi entropy of order 2 that determines the leakage (see [3] for a similar observation). However, the weaker bound proved above suffices for our case, as it does for many other cases (*cf.* [24]).

C. A secret key agreement protocol requiring 1-bit feedback

In this section, we present a secret key agreement protocol which requires only 1-bit of feedback for generating an (ϵ, δ) -SK, in the special case when Z is a constant. The main component is a high secrecy protocol which achieves arbitrarily high secrecy and required reliability. In contrast to Protocol 1, which relied on slicing the spectrum of $P_{X|Y}$, the high secrecy protocol is based on slicing the spectrum of P_X . Since the party observing X can determine the corresponding slice index, feedback is not needed and one-way communication suffices. We then convert this high secrecy protocol into a high reliability protocol, using a 1-bit feedback. The required protocol for generating an (ϵ, δ) -SK is obtained by randomizing between the high secrecy and the high reliability protocols. This protocol appeared in a conference version containing some of the results of this paper [16], but was discovered independently by [8] in a slightly different setting. Note that this is a different approach from the one used in Section IV where a high reliability protocol was constructed and Proposition 1 was invoked to obtain a high secrecy protocol.

Description of the high secrecy protocol. We now describe our protocol formally. The information reconciliation step of our protocol relies on a single-shot version of the classical Slepian-Wolf theorem [27] in distributed source coding for two sources [20], [10, Lemma 7.2.1] (see, also, [18]). We need a slight modification of the standard version – the encoder is still a random binning but for decoding, instead of using a “typical-set” decoder for the underlying distribution, we use a mismatched typical-set decoder. We provide a proof for completeness.

Lemma 13 (Slepian-Wolf Coding). *Given two distributions P_{XY} and Q_{XY} on $\mathcal{X} \times \mathcal{Y}$, for every $\gamma > 0$ there exists a code (e, d) of size M with encoder $e : \mathcal{X} \rightarrow \{1, \dots, M\}$, and a decoder $d : \{1, \dots, M\} \times \mathcal{Y} \rightarrow \mathcal{X}$, such that*

$$P_{XY}(\{(x, y) : x \neq d(e(x), y)\}) \leq P_{XY}(\{(x, y) : -\log Q_{X|Y}(x|y) \geq \log M - \gamma\}) + 2^{-\gamma}.$$

Proof. We use a random encoder given by random binning, *i.e.*, for each $x \in \mathcal{X}$, we independently randomly assign $i = 1, \dots, M$. For the decoder, we use a typicality-like argument, but instead of using the standard typical set defined via $P_{X|Y}$, we use the mismatched typical-set

$$\mathcal{T}_{Q_{X|Y}} := \{(x, y) : -\log Q_{X|Y}(x|y) < \log M - \gamma\}.$$

Then, upon receiving $i \in \{1, \dots, M\}$, the decoder outputs \hat{x} if there exists a unique \hat{x} satisfying $e(\hat{x}) = i$ and $(\hat{x}, y) \in \mathcal{T}_{Q_{X|Y}}$. An error occur if $(X, Y) \notin \mathcal{T}_{Q_{X|Y}}$ or there exists $\tilde{x} \neq X$ such that $(\tilde{x}, Y) \in \mathcal{T}_{Q_{X|Y}}$ and $e(\tilde{x}) = e(X)$. The former error event occurs with probability

$$P_{XY}(\{(x, y) : -\log Q_{X|Y}(x|y) \geq \log M - \gamma\}).$$

The probability of the second error event averaged over the random binning is bounded as

$$\begin{aligned} & \mathbb{E} \left[\mathbb{P}(\exists \tilde{x} \neq X \text{ s.t. } e(\tilde{x}) = e(X), (\tilde{x}, Y) \in \mathcal{T}_{Q_{X|Y}}) \right] \\ & \leq \sum_{x,y} P_{XY}(x, y) \mathbb{E} \left[\sum_{\tilde{x} \neq x} \mathbb{1}(e(\tilde{x}) = e(x)) \cdot \mathbb{1}((\tilde{x}, y) \in \mathcal{T}_{Q_{X|Y}}) \right] \\ & = \sum_{x,y} P_{XY}(x, y) \sum_{\tilde{x} \neq x} \frac{1}{M} \mathbb{1}((\tilde{x}, y) \in \mathcal{T}_{Q_{X|Y}}) \\ & \leq \sum_{x,y} P_{XY}(x, y) 2^{-\gamma} \\ & = 2^{-\gamma}, \end{aligned}$$

where the expectation is over the random encoder e . The first inequality above is by the union bound, the first equality is a property of random binning, and the second inequality follows from

$$|\{x : (x, y) \in \mathcal{T}_{Q_{X|Y}}\}| \leq M2^{-\gamma} \quad \forall y \in \mathcal{Y}.$$

Thus, there exists a code (e, d) satisfying the desired bound. \square

We are now in a position to describe our protocol, which is based on slicing the spectrum of P_X . We first slice the spectrum of P_X into $L + 1$ parts. Specifically, for $1 \leq i \leq L$, let $\lambda_i = \lambda_{\min} + (i - 1)\Delta$

and define

$$\mathcal{X}_i := \{x : \lambda_i \leq -\log P_X(x) < \lambda_i + \Delta\}. \quad (30)$$

We also define

$$\mathcal{X}_0 := \{(x, y) : -\log P_X(x) \geq \lambda_{\max} \text{ or } -\log P_X(x) < \lambda_{\min}\}. \quad (31)$$

Denote by J the RV such that the event $\{J = j\}$ corresponds to \mathcal{X}_j , $0 \leq j \leq L$. We divide the indices $0 \leq j \leq L$ into “good” indices \mathcal{I}_g and the “bad” indices $\mathcal{I}_b = \mathcal{I}_g^c$, where

$$\mathcal{I}_g = \left\{ j : j > 0 \text{ and } P_J(j) \geq \frac{1}{L^2} \right\}.$$

Denote by P_j the conditional distribution of X, Y given $J = j$, *i.e.*,

$$P_j(x, y) = \frac{P_{XY}(x, y)}{P_X(\mathcal{X}_j)} \mathbf{1}(x \in \mathcal{X}_j), \quad x \in \mathcal{X}, y \in \mathcal{Y}, 0 \leq j \leq L.$$

Note that J is a function of X and can be computed by the first party, *i.e.*, the party observing X . In our protocol below, the first party computes J and sends it to the second party as public communication. If $J \in \mathcal{I}_b$, the protocol declares a reconciliation error and aborts. Otherwise, the protocol generates a secret key conditioned on the event \mathcal{X}_J .

For $1 \leq j \leq L$, let (e_j, d_j) be the Slepian-Wolf code of Lemma 13 for $P_{XY} = P_j$ and $Q_{XY} = P_{XY}$. Further, let \mathcal{F} be a 2-universal family of mappings $f : \mathcal{X} \rightarrow \mathcal{K}$, and let S be random seed such that f_S denotes a randomly chosen member of \mathcal{F} .

Our secret key agreement protocol is given in Protocol 2.

Protocol 2: High secrecy protocol

Input: Observations X and Y

Output: Secret key estimates K_x and K_y

Information reconciliation

First party (observing X) finds the index $J \in \{0, 1, \dots, L\}$ such that $X \in \mathcal{X}_J$

if $J \in \mathcal{I}_b$ **then**

| The protocol declares an error and aborts

else

| First party sends $(J, e_J(X))$ to the second party

| Second party computes $\hat{X} = d_J(Y, e_J(X))$

| Privacy amplification

| First party generates the random seed S and sends it to the second party using public communication

| First party generates the secret key $K_x = K = f_S(X)$

| The second party generates the estimate K_y of K as $K_y = f_S(\hat{X})$

Performance bounds for Protocol 2. The next result shows that Protocol 2 attains arbitrary high secrecy and required reliability.

Theorem 14. For every $\gamma > 0$ and $0 \leq \lambda \leq \lambda_{\min}$, Protocol 2 yields an (ϵ, δ) -SK K taking values in \mathcal{K} with

$$\begin{aligned} \epsilon &\leq \mathbb{P}(i_{XY}(X, Y) \leq \lambda + \gamma + \Delta) + \mathbb{P}_{XY}(\mathcal{X}_0) + 2^{-\gamma} + \frac{1}{L}, \\ \delta &\leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - 2 \log L)}}, \end{aligned}$$

where, with $\lambda_{\max} = \lambda_{\min} + L\Delta$, \mathcal{X}_0 is given by (31).

Proof. To bound the error in information reconciliation, note that for all $j \in \mathcal{I}_g$ by Lemma 13 with $\mathbb{P}_{XY} = \mathbb{P}_j$ and $\mathbb{Q}_{XY} = \mathbb{P}_{XY}$

$$\begin{aligned} &\mathbb{P}_j(\{(x, y) : x \neq d_j(e_j(x), y)\}) - 2^{-\gamma} \\ &\leq \mathbb{P}_j(\{(x, y) : -\log \mathbb{P}_{X|Y}(x | y) \geq \log M_j - \gamma\}) \\ &= \mathbb{P}_j\left(\left\{(x, y) : -\log \mathbb{P}_X(x) - i_{XY}(x, y) \geq \log M_j - \gamma\right\}\right) \\ &\leq \mathbb{P}_j\left(\left\{(x, y) : \lambda_j + \Delta - i_{XY}(x, y) \geq \log M_j - \gamma\right\}\right), \end{aligned}$$

where the previous inequality uses the definition of P_j and (30). On choosing

$$\log M_j = \lambda_j - \lambda,$$

we get

$$P_j(\{(x, y) : x \neq d_j(e_j(x), y)\}) \leq P_j(\{(x, y) : i_{XY}(x, y) \leq \lambda + \gamma + \Delta\}) + 2^{-\gamma}.$$

An error in information reconciliation occurs if either $J \notin \mathcal{I}_g$ or if $j \in \mathcal{I}_g$ and $X \neq d_j(e_j(X), Y)$. From the bound above

$$\begin{aligned} \epsilon &\leq P_{XY}(J \notin \mathcal{I}_g) + 2^{-\gamma} + \sum_{j \in \mathcal{I}_g} P_J(j) P_j(\{(x, y) : i_{XY}(x, y) \leq \lambda + \gamma + \Delta\}) \\ &\leq P_{XY}(J \notin \mathcal{I}_g) + 2^{-\gamma} + P_{XY}(\{(x, y) : i_{XY}(x, y) \leq \lambda + \gamma + \Delta\}), \end{aligned}$$

which using

$$P_J(\mathcal{I}_b) = \sum_{j \in \mathcal{I}_b} P_J(j) \leq P_J(0) + \frac{1}{L}$$

gives

$$\epsilon \leq P_X(\mathcal{X}_0) + 2^{-\gamma} + \frac{1}{L} + P_{XY}(\{(x, y) : i_{XY}(x, y) \leq \lambda + \gamma + \Delta\}),$$

proving the reliability bound of the theorem.

We proceed to secrecy analysis. Note that the protocol only defines the secret key for the case $J \in \mathcal{I}_g$. For concreteness, let

$$K = \begin{cases} f_S(X), & J \in \mathcal{I}_g, \\ \text{unif}(\mathcal{K}), & \text{otherwise,} \end{cases}$$

K is perfectly secure when $J \in \mathcal{I}_b$. Denoting the communication $(J, e_J(X))$ by \mathbf{F} , we get

$$\begin{aligned} &\|P_{K\mathbf{F}S} - P_{\text{unif}}P_{\mathbf{F}S}\|_1 \\ &= \sum_{j \notin \mathcal{I}_g} P_J(j) \cdot 0 + \sum_{j \in \mathcal{I}_g} P_J(j) \cdot \|P_{Ke_J(X)S|J=j} - P_{\text{unif}}P_{e_J(X)S|J=j}\|_1. \end{aligned} \quad (32)$$

To bound $\|P_{Ke_J(X)S|J=j} - P_{\text{unif}}P_{e_J(X)S|J=j}\|_1$, denote by $P_{j,X}$ the marginal on \mathcal{X} induced by P_j .

Note that for each $j \in \mathcal{I}_g$

$$\begin{aligned} -\log P_{j,X}(x) &= -\log \frac{P_X(x)}{P_X(\mathcal{X}_j)} \\ &\geq \lambda_j - 2 \log L, \end{aligned}$$

where the last inequality uses the definition of \mathcal{X}_j and \mathcal{I}_g . It follows that

$$H_{\min}(P_{j,X}) \geq \lambda_j - 2 \log L.$$

Therefore, upon noting that S is independent of (X, Y, J) even upon conditioning on $J = j$, for each $j \in \mathcal{I}_g$ an application of Lemma 4 implies that

$$\begin{aligned} \|\mathbb{P}_{K e_j(X)S|J=j} - \mathbb{P}_{\text{unif}} \mathbb{P}_{e_j(X)S|J=j}\|_1 &\leq \frac{1}{2} \sqrt{|\mathcal{K}| M_j 2^{-H_{\min}(P_{j,X})}} \\ &\leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - 2 \log L)}}, \quad j \in \mathcal{I}_g, \end{aligned}$$

which together with (32) gives

$$\|\mathbb{P}_{KFS} - \mathbb{P}_{\text{unif}} \mathbb{P}_{FS}\|_1 \leq \frac{1}{2} \sqrt{|\mathcal{K}| 2^{-(\lambda - 2 \log L)}},$$

which in turn proves the secrecy bound claimed in the theorem. \square

From high secrecy protocol to a high reliability protocol. By Theorem 14, the secrecy parameter δ of Protocol 2 can be made small by choosing $\log |\mathcal{K}| \approx \lambda$, but its reliability parameter ϵ is limited by the tail-probability $\mathbb{P}(i_{XY}(X, Y) \leq \lambda)$. Thus, in contrast to Protocol 1, Protocol 2 constitutes a high secrecy protocol. Note that while any high reliability protocol can be converted into a high secrecy protocol using Proposition 1, it is unclear if a high secrecy protocol can be converted to a high reliability protocol in general. However, high secrecy Protocol 2 can be converted into a high reliability protocol as follows: The second party upon decoding X computes the indicator of the error event $\mathcal{E}_j := \{-\log P_{X|Y}(X | Y) \geq \log M_j - \gamma\}$ and sends it back to the first party. If \mathcal{E}_j doesn't occur, the secret key K is as in the protocol above. Otherwise, K is chosen to be a constant. For this modified secret key, the event \mathcal{E}_j is accounted for in the secrecy parameter δ and not in ϵ as earlier. Thus, Theorem 14 holds for the modified secret key where the leading term $\mathbb{P}(i_{XY}(X, Y) \leq \lambda + \gamma + \Delta)$ is moved from the upper bound on ϵ to that on δ , and the resulting protocol has high reliability. Furthermore, the high reliability protocol uses just 1-bit of feedback from the second party to the first.

Finally, a protocol for generating an arbitrary (ϵ, δ) -SK can be obtained by a hybrid use of the high reliability and high secrecy protocols as in Proposition 2.

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