Transmission Capacity of Wireless Ad Hoc Networks

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Workshop on Models and Protocols for Mobile Ad-Hoc Networks

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Motivation

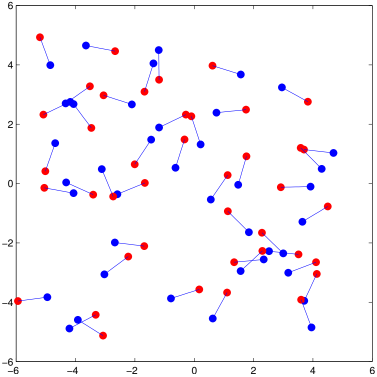
- A set of links (TX and RX) are randomly distributed in $\mathbb{R}^2$.
- A MAC protocol activates a subset of these links
  - ALOHA, CSMA, TDMA
- All nodes transmit at the same power
- A scheduled link succeeds if its SIR $> \theta$

Questions

1. How to quantify the efficiency (packing) of the MAC protocol?
2. Better signal processing or better MAC?

Dipole model

- The links are distributed as a stationary point process $\Phi \subset \mathbb{R}^2$ of unit
- Each transmitter has a receiver at a distance d in a random direction
- Path loss function is denoted by $\ell(x)$. Eg. $\ell(x) = \|x\|^{-\alpha}$
- A subset of these nodes $\Phi_t \subset \Phi$ are activated by a MAC protocol
  - Density: λ_t

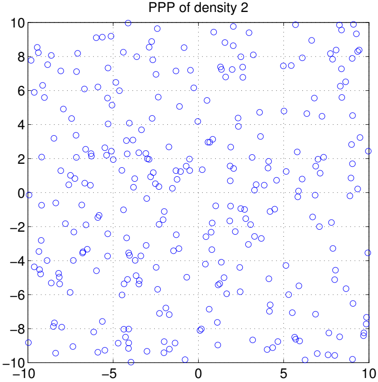


Quantify performance of MAC and its interaction with the signal processing

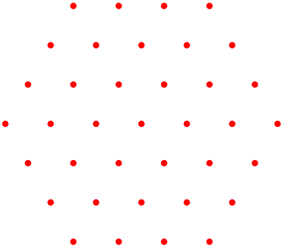
Some models for spatial locations of the links

Spatial poisson point process

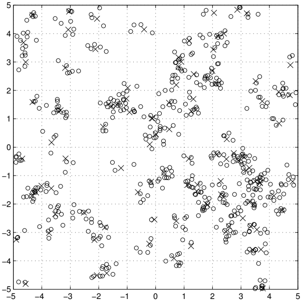
1. The most widely used model for spatial locations of nodes
   - Most amicable to analysis
   - "Gaussian of point processes"
2. No dependence between node locations
3. Random number of nodes
4. Can be defined on the entire plane



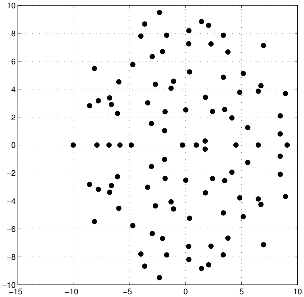
Other interesting examples



Hexagonal lattice



Thomas cluster process



Determinantal point process (eigenvalues of a Gaussian matrix)

Examples of MAC schemes

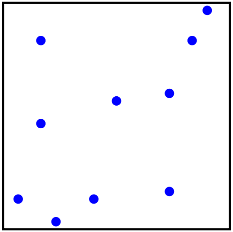
ALOHA

Let Φ be a PP of density λ

1. A node (link) $x \in \Phi$ is scheduled with probability p independent of other nodes.
2. The transmitter process Φ_t is stationary with density λp.

CSMA model: Matern technique

1. Begin with a PP Φ of density λ.
2. To each $x \in \Phi$, associate a mark $m_x \sim U[0,1]$ independent of every other point.
3. A node $x \in \Phi$ selected if it has the lowest mark among all the points in the ball $B(x, R)$.



$$\Psi = \{y : y \in \Phi, m_y \leq m_x, \forall x \in B(y, R) \cap \Phi\}$$

A minimum distance process for modelling CSMA MAC.

CSMA model: Matern technique

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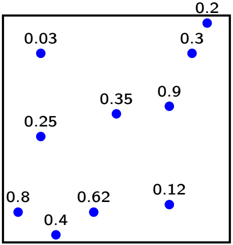

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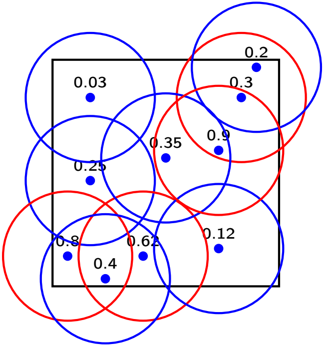

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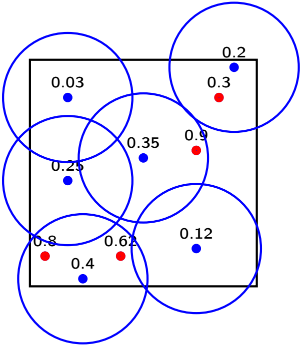
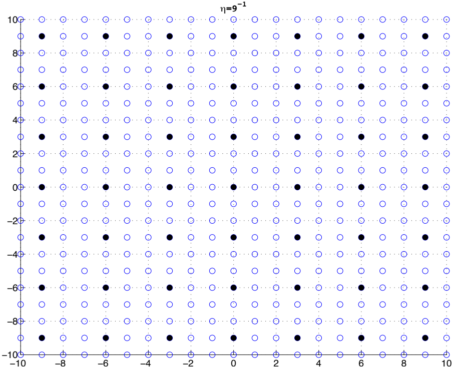


A minimum distance process for modelling CSMA MAC.

m-Phase TDMA



Signal processing techniques

- TX has N_t transmit antenna
- RX has N_r receive antenna
- MRC, spatial multiplexing....

Link formation probability

Signal-to-interference ratio

$$\mathrm{SIR}(x) = \frac{h_{xr(x)}\ell(d)}{\sum_{y\in\Phi_t} h_{yr(x)}\ell(y - r(x))}$$

The success probability of a *typical link*

$$P_c(\theta,\lambda) = \mathbb{P}^{!o}(\mathrm{SIR}(o) > \theta)$$

Reduced Palm probability:

- picking a link at random
- average over all the links
- conditioning on a point at the origin (but not counting it)

Metric: Transmission capacity(TC)

Let $\epsilon \in (0,1)$. TC is defined as

$$TC(\epsilon) = (1-\epsilon)\max\{\lambda : \mathrm{p}_s(\theta,\lambda) > 1-\epsilon\}$$

1. TC measures the maximum spatial intensity of successful transmissions per unit area for a given success probabiltiy.
2. Can be related to area spectral efficiency (ASE) by multiplying with $\log_2(1+\theta)$.

We are interested in the behaviour (scaling) of $TC(\epsilon)$ when $\epsilon \to 0$

ALOHA MAC

Node distribution: $\Phi \sim \text{PP}(1)$
MAC: ALOHA with parameter p.
Φ_t is a stationary PP with density p.

$$
\begin{aligned}
p_s(\theta, \lambda) &= \mathbb{P}^{!o}(\text{SIR}(o, r(o)) > \theta) \\
&= \mathbb{P}^{!o}\left(\frac{S\ell(d)}{I_t} > \theta\right) = \mathbb{E}^{!o}\left[F_s\left(\theta \frac{I_t}{\ell(d)}\right)\right]
\end{aligned}
$$

Here $I_t = \sum_{x \in \Phi} \mathbf{1}(x \text{is selected}) h_x \ell(x - r(o))$

Basic idea: 2 node network

Let $Y_i = \frac{\theta h_{x_i}}{\ell(d)} l(x_i - r(o)), i = 1, 2.$ Then

$$\hat{I}_t = Y_1 \mathbf{1}(1 \text{ is on}) + Y_2 \mathbf{1}(2 \text{ is on}).$$

Hence averaging over the ALOHA selection

$$\begin{aligned}\mathbb{E}_p[F_s(\hat{I}_t)] &= (1-p)^2 F_s(0) + p(1-p)(F_s(Y_1) + F_s(Y_2)) + p^2 F_s(Y_1 + Y_2) \\ &\sim 1 - p\sum_{i=1}^{2}(1 - F_s(Y_i)) + o(p)\end{aligned}$$

ALOHA: Asymptotic success probability

Theorem (RK-MH-JA)

When ALOHA is used as the MAC protocol, the density of Φ_t is $\lambda_t = p$ and

$$P_s(\theta, \lambda_t) \sim 1 - \lambda_t \gamma_{ALOHA}, \ \lambda_t \to 0,$$

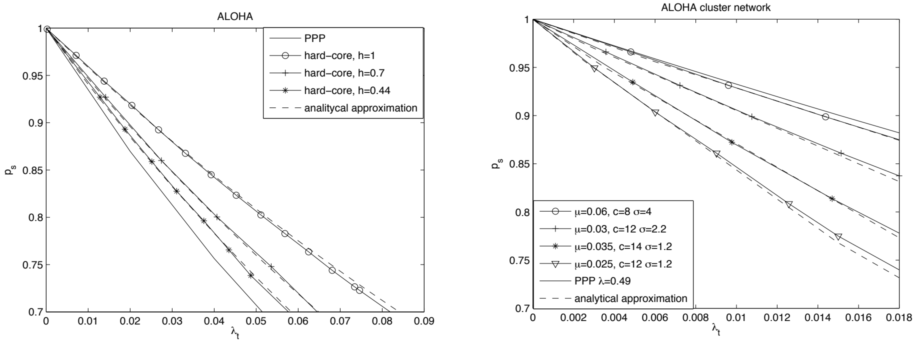
where

$$\gamma_{ALOHA} = \int_{\mathbb{R}^2} \left[1 - E_h F_s\left(h \frac{\theta \ell(x - r(o))}{\ell(d)}\right)\right] \rho^{(2)}(x) \mathrm{d}x$$

$$p_s(\theta, \lambda) = \mathbb{E}^{!o}\left[F_s\left(\frac{\theta}{\ell(d)} I_t\right)\right]$$

$$\sim 1 - p\mathbb{E}^{!o}\left[\sum_{x \in \Phi} 1 - F_s\left(\frac{\theta}{\ell(d)} h_x \ell(x - r(o))\right)\right] + o(p)$$

How good is the approximation



ALOHA on a hard-core network; Right: ALOHA on a clustered network

ALOHA: Transmission capacity

$$TC_{\text{ALOHA}}(\epsilon) \sim \frac{\epsilon}{\gamma_{\text{ALOHA}}}, \quad \epsilon \to 0$$

- Scales linearly with ϵ
- Only γ_{ALOHA} depends on $F_s(.)$, h and Φ

γ_{ALOHA} for a PPP with $\ell(x) = \|x\|^{-\alpha}$, $\delta \triangleq 2/\alpha$

1. Nakagami-m fading with $N_t = N_r = 1$

$$\gamma_{\text{ALOHA}} = d^2 \frac{\pi\theta^\delta \Gamma(m-\delta)\Gamma(m+\delta)}{\Gamma(m)^2} \sim \Theta(1)$$

2. Receiver beamforming: $N_t = 1$, $N_r = m$

$$\gamma_{\text{ALOHA}} = d^2 \frac{\pi\theta^\delta \Gamma(m-\delta)\Gamma(1+\delta)}{\Gamma(m)} \sim \Theta(m^{-\delta})$$

Sphere packing interpretation of TC

$$TC(\epsilon) \sim \frac{\epsilon}{d^2 C(\delta)} = \frac{1}{\pi \left(d\sqrt{\frac{C(\delta)}{\pi\epsilon}} \right)^2}.$$

Interpretation (heuristic)

Hence each transmission approximately requires an interference free disc of radius

$$R = d\sqrt{\frac{C(\delta)}{\pi\epsilon}}$$

- The disc radius increases as $\frac{1}{\sqrt{\epsilon}}$.
- The disc radius decreases with increasing α
  - Higher path loss exponent $\rightarrow$ better packing.

CSMA MAC

Node distribution: $\Phi \sim \text{PPP}(1)$
MAC: CSMA with radius R.
Φ_t is a stationary PP with density $\lambda_t = \frac{1-\exp(-\pi R^2)}{\pi R^2}$.

$$
\begin{aligned}
p_s(\theta, \lambda) &= \mathbb{P}^{!o}(\text{SIR}(o, r(o)) > \theta) \\
&= \mathbb{P}^{!o}\left(\frac{S\ell(d)}{I_t} > \theta\right) = \mathbb{E}^{!o}\left[F_s\left(\theta \frac{I_t}{\ell(d)}\right)\right]
\end{aligned}
$$

Here $I_t = \sum_{x \in \Phi} \mathbf{1}(x \text{ is selected}) h_x \ell(x - r(o))$

Basic idea

- Minimum distance between nodes $\mathbb{E}[I] < \infty$. Not true for ALOHA
- Taylor series of CCDF of the desired signal

$$F_s(x) = 1 - c_0 x^{\nu} + o(x^{\nu}), x \to 0.$$

- Average nearest interferer distance $\approx 1/\sqrt{\lambda_t}$.
-

$$F_s(I) \approx 1 - c_0 \lambda^{\alpha\nu/2}.$$

CSMA: Asymptotic success probability and TC

Theorem

For a Matern CSMA process,

$$P_s(\theta, \lambda_t) \sim 1 - c_0 \pi \left(\frac{\theta}{\ell(d)} \right)^{\nu} \lambda_t^{\nu/\delta} A_I, \lambda_t \to 0.$$

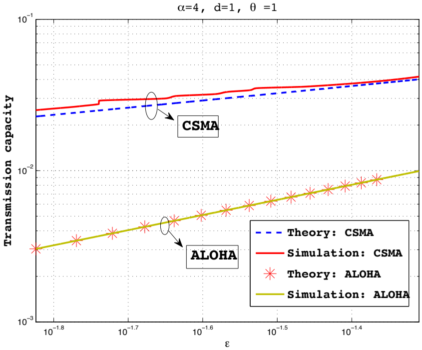
- A_I is known and depends on the higher order moment densities ($\nu + 1$ of them) of the Matern process
- Difficult to evaluate except for $\nu = 1$

Transmission capacity

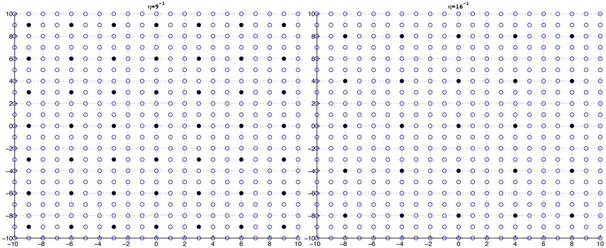
$$TC_{\text{CSMA}}(\epsilon) \sim \frac{\epsilon^{\delta/\nu}}{(c_o \pi A_I)^{\delta/\nu} \theta^{\delta} \ell(d)^{-\delta}}, \quad \epsilon \to 0$$

Observations

- Better TC $\Theta(\epsilon^{\delta/\nu})$
- Interacts well with signal processing techniques
  - Nakagami-m: $\nu = m$
  - MRC: $\nu = N_r$



m-phase TDMA with $F_s(x) = \exp(-x)$



Left: $\lambda_t = 1/9$; Right: $\lambda_t = 1/16$.

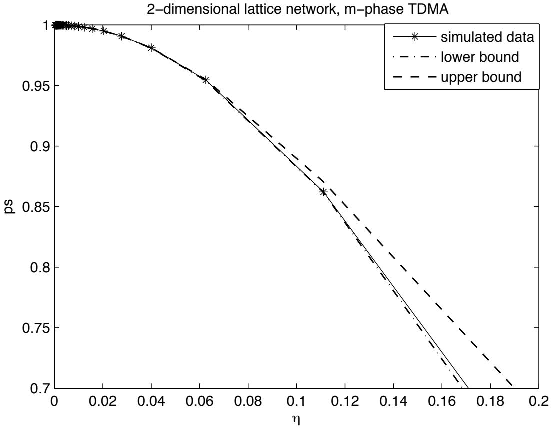
$$p_c(\theta, \lambda_t) \sim 1 - \gamma_{\text{TDMA}} \lambda_t^{1/\delta},$$

where

$$\gamma_{\text{TDMA}} = \frac{\theta \xi(\delta^{-1}, 0)[\xi(\delta^{-1}, 1/4) - \xi(\delta^{-1}, 3/4)]}{\ell(d) 2^{2(\delta^{-1}-1)}},$$

and $\xi(s, b)$ is the generalized Riemann zeta function.

m-phase TDMA


2−dimensional lattice network, m−phase TDMA
simulated data
lower bound
upper bound
ps
η
1
0.95
0.9
0.85
0.8
0.75
0.7
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2

Reasonable MAC

A.1 $\lim_{\lambda \to 0} \mathbb{E}^{!o}(\text{Points of } \Phi \text{ in } B(o, \lambda^{-a})) = 0$ for all $a < 1/2$.

A.2 There exists some constant R such that $\lim_{\lambda \to 0} \mathbb{E}^{!o}(\text{Points of } \Phi \text{ in } B(o, R\lambda^{-1/2})) > 0$.

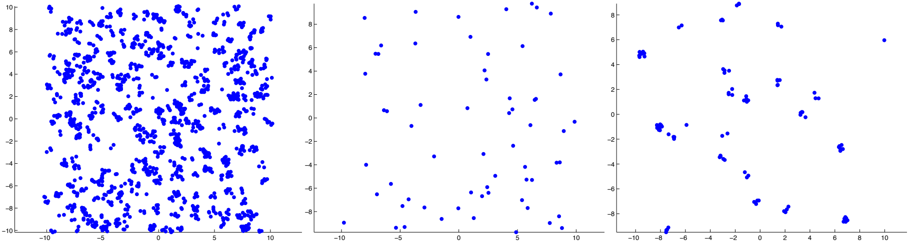
Definition

Any MAC scheme which satisfies A.1 and A.2 is a reasonable MAC scheme

Example: PPP with ALOHA is a reasonable MAC scheme.

An unreasonable MAC scheme

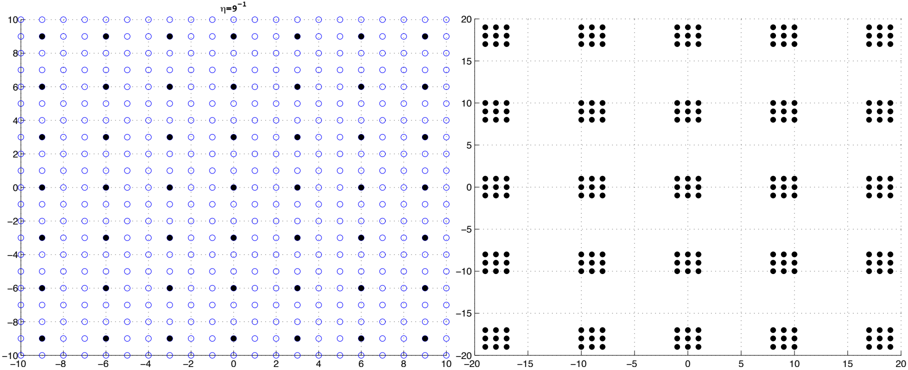
Consider a cluster process with average number of clusters per unit area as λ_p and nodes per cluster as $\bar{c}$.



$$\lambda_p = 1, \bar{c} = 15, n = 25$$

- Using ALOHA select a node with probability n^{-1}.
  Nodes per cluster: $\bar{c}n^{-1}$, Density: $\bar{c}\lambda_p n^{-1}$.
- Allow a cluster to transmit with probability n^{-1}.
  Cluster density: $\lambda_p n^{-1}$, Density: $\bar{c}\lambda_p n^{-1}$.

m-Phase TDMA



Bounds on scaling exponent

For a reasonable MAC

$$p_s(\Theta, \lambda_t) \sim 1 - \gamma\lambda_t^{\kappa}, \quad \lambda_t \to 0$$

κ: Interference scaling exponent. γ: Spatial contention parameter

Theorem

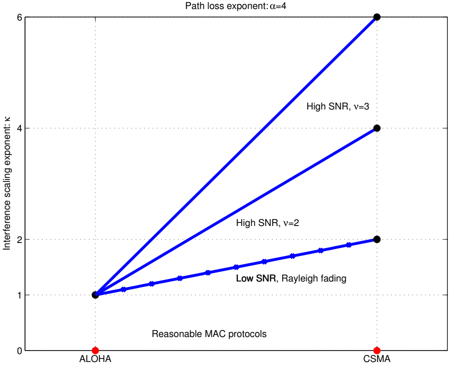
For any reasonable MAC protocol,

$$1 \leq \kappa \leq \frac{\nu}{\delta},$$

- $\kappa = 1$: ALOHA
- $\kappa = \frac{\nu}{\delta}$: CSMA

Interference scaling exponent κ

$$\mathrm{F}_s\left(\sigma^2 + x\right) = \mathrm{F}_s\left(\sigma^2\right) + C_0 x^{\nu} + o(x^{\nu}), x \to 0$$



Low SNR, $\sigma^2 \neq 0$

$\nu = 1$: $1 \leq \kappa \leq \alpha/2$

High SNR, $\sigma^2 \approx 0$

- SISO, Rayleigh fading, $\nu = 1$, $1 \leq \kappa \leq \alpha/2$
- High SNR: Nakagami-m, receiver beamforming (m antenna), $\nu = m$, $1 \leq \kappa \leq m\alpha/2$

$1 \leq \kappa \leq \frac{\nu\alpha}{2}$, exponent κ depends on the diversity order of the receiver

Key results

1. For any reasonable MAC scheme, the interference scaling exponent $1 \leq \kappa \leq \nu/\delta$
2. ALOHA always leads to $\kappa = 1$
3. CSMA leads to $\kappa = \nu/\delta$
4. Interference cancellation (CSMA) is important to take advantage of advanced signal processing techniques
5. Transmission capacity provides a clean framework for the analysis of networks