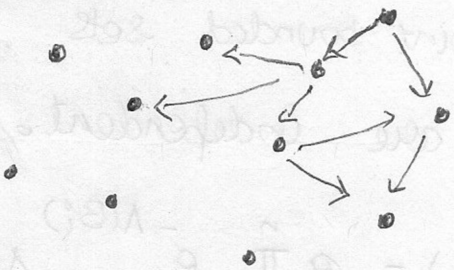
16/10/19

INTRODUCTION TO POINT PROCESSES.



PERFORMANCE MEASURES
$=$
F (points).

Questions - Mathematical model for random point sets?

Evaluate functionals of point sets?

(Performance measures).

$\Phi = \{X_i\}_{i \geq 1}$; $X_i \in \mathbb{R}^d$, random variables in $\mathbb{R}^d$

* Not necessarily independent or identically distributed.

* Countably many at the most

* $\Phi(B) := \# \{ i : X_i \in B \}$; $\Phi(B) < \infty$ a.s. $\forall$ bounded $B \subseteq \mathbb{R}^d$. (locally finite / Radon)

* (SIMPLE) : $X_i \neq X_j \quad \forall i \neq j$.

EXAMPLES: (LATTICE): $\Phi = \mathbb{Z}^d$; $d \geq 1$

(BINOMIAL POINT PROCESS) $\Phi = \{X_1, \dots, X_n\}$

$X_1, \dots, X_n$ - i.i.d. distributed in $\mathbb{R}^d$ with distrib^n $f(\cdot)$.

$$p_B := P(X_i \in B) = \int_B f(x)\,dx.$$

$$\Phi(B) = \sum_{i=1}^{n} 1[X_i \in B] \overset{d}{=} Bin(n, p_B). \text{ (Binomially distributed)}$$

Common choices of f = $f(x) = 1\left[x \in [-1/2, 1/2]^d\right]$

(Uniform Binomial point process on the cube).

POISSON POINT PROCESS: f - ≥ 0 function on $\mathbb{R}^d$ &

$\Lambda(B) = \int_B f(x)\,dx < \infty$. Φ - is a Poisson point process with intensity $f(\cdot)$ if

(i) $\Phi(B) \overset{d}{=} \text{Poisson}(\Lambda(B))$

(ii) if $B_1, \ldots, B_n$ are disjoint bounded sets then $\Phi(B_1), \ldots, \Phi(B_n)$ are independent.

$$P(\Phi(B_1) = k_1, \ldots, \Phi(B_n) = k_n) = \prod_{i=1}^{n} e^{-\Lambda(B_i)} \frac{\Lambda(B_i)^{k_i}}{k_i!}$$

HOMOGENEOUS PPP: P_λ - $f(x) = \lambda$

INHOMOGENEOUS PPP: P_f - $f(x) > 0$ if $x \in A$, some compact, convex set & $\int_{\mathbb{R}^d} f(x) = 1$

& $0 < \inf_{x \in A} f(x) \leq \sup_{x \in A} f(x) < \infty$.

OTHER ~~RECENT~~ MODELS (From statistical physics)

M - Random matrix - Φ - Eigen-values of M.

F - Random function - Φ - Zeros of F.

MANJUNATH KRISHNAPUR

OPERATIONS ON POINT PROCESSES: (Not simple)

Φ_1, Φ_2 - two point processes; $\Phi_1 + \Phi_2$ - point process

$\shortparallel$

$\Phi_1 \cup \Phi_2$

(SUPER-POSITION)

TRANSFORMATION: $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $T^{-1}(B)$ is bdd for B bdd.

then $T\Phi = \{Tx_i : x_i \in \Phi\}$ is also a point process.

(Not simple).

THINNING: $\{\xi_x\}_{x \in \mathbb{R}^d}$ – Bernoulli random variables

i.e., $\xi_x \in \{0,1\}$ Random variables.

$$\Phi_\xi := \{ \xi_{X_i} X_i \}_{i \geq 1}$$ – i.e., keep points with $\xi_{X_i} = 1$.

USEFUL Eg: $\xi_x = 1[\Phi(B_r(x)) = 0]$.

$$\Rightarrow \quad \Phi_\xi := \{ X_i \, 1[\Phi(B_r(X_i)) = 0] \}$$ HARD-CORE POINT PROCESSES.

i.e., $X_i, X_j \in \Phi_\xi \Rightarrow |X_i - X_j| > r$.

OFT-USED CLASSES.

STATIONARY: $\Phi + x = \{x + X_i\}$

– $\Phi + x \overset{d}{=} \Phi \quad \forall \; x \in \mathbb{R}^d$.

$\alpha(B) = \mathbb{E}[\Phi(B)]$ – Intensity / mean measure.

Stationarity $\Rightarrow \alpha(B) = \alpha(B+x) \quad \forall \; x$.

$\Rightarrow \alpha(B) = \lambda |B|$. on λ

SUPER-POSITION: $\alpha_\Phi(B) = \alpha_1(B) + \alpha_2(B)$. [Linearity of Expectation]

TRANSFORMATION: $\alpha_T(B) = \alpha(T^{-1}(B))$

~~THINNING~~

Suppose $B \ni |B| = 0 \Rightarrow \alpha(B) = 0 \Rightarrow \Phi(B) = 0$ a.s.

For eg. $B = \{x\}$ – i.e., in a fixed set of volume 0, prob of any point falling is 0.

$\lambda = 0 \Rightarrow \Phi(B) = 0 \; \forall \; B$. Let $W_n = [-n, n]^d$ $\Phi(W_n) = 0 \; \forall n$

$$\Phi(\mathbb{R}^d) = \bigcup_n \Phi(W_n) = 0 \text{ a.s.}$$

$\lambda > 0$ $\quad \alpha(W_n) = \lambda n^d \to \infty .$

$\Rightarrow P(\Phi(\mathbb{R}^d) = \infty) > 0 .$

POISSON / BINOMIAL POINT PROCESS:

N_n – Poisson(n) random variable

$$P(|N_n - n| > n^{\gamma}) \leq C e^{-c n^{2\gamma - 1}} , \quad \gamma > 1/2 .$$

For eg. $\gamma = 3/4$

$$n - n^{3/4} \leq N_n \leq n + n^{3/4}$$

$P_{nf(\cdot)} = \{X_1, \ldots, X_{N_n}\}$ $\quad X_i$ – i.i.d $f(\cdot)$ distributed & independent of N_n

$$P_{nf}(B) = \sum_{i=1}^{N_n} 1[X_i \in B] \overset{d}{=} \text{Poi}(n \Lambda(B))$$

→ Let $\{\xi_i\}_{i=1}^{\infty}$ be i.i.d. random variables in $\{1, \ldots, k\}$ $p(1), \ldots, p(k)$

then $\#\{i \leq N_n : \xi_i = j\} \overset{d}{=} \text{Poi}(n p(j))$ $\quad 1 \leq j \leq k$.

Proves Poisson point process.

→ $P_{nf(\cdot)} \mid N_n = n \overset{d}{=} \{X_1, \ldots, X_n\}$. i.e., Binomial point process.

→ If $N_1 \sim \text{Poi}(\lambda)$, $N_2 \sim \text{Poi}(\mu)$ & N_1, N_2 are independent

then $N_1 + N_2 \sim \text{Poi}(\lambda + \mu)$

$\Rightarrow$ $P_{nf(\cdot)}$ & $P_{mg(\cdot)}$ independent Poisson point processes

then $P_{nf(\cdot)} + P_{mg(\cdot)} \overset{d}{=} P_{nf(\cdot) + mg(\cdot)}$.

Extended to locally integrable $f(\cdot)$ & $g(\cdot)$.

UNIQUENESS: P_1 & P_2 – Poisson point processes such that $E[P_1(B)] = E[P_2(B)]$ $\forall$ bounded B i.e., $\int_B f_1(x)\,dx = \int_B f_2(x)\,dx$

$\Rightarrow f_1(x) = f_2(x)$ a.e. & $P_1 \overset{d}{=} P_2$.

i.e., Intensity measure determines the point process.

Not true in general.

TRANSFORMATION: In general, hard to express. Some cases via Jacobian et al. But we'll see specific cases later.

THINNING: $\{\xi_x\}$ – i.i.d. Bernoulli random variables 1 w.p. $p(x)$

$$\alpha_\Phi^T(B) = \mathbb{E}\Big[\sum_{i=1}^{P(B)} \xi_{X_i}\Big]$$

$X_1, \dots, X_{P(B)}$ – independent & distributed $\frac{f(\cdot)}{\Lambda(B)}$

$X_1, \dots, X_{P(B)}$ are independent of $P(B)$.

$$= \Lambda(B)\, P(\xi_{X_i} = 1)$$

$$= \Lambda(B) \cdot \int_B p(x) \frac{f(x)}{\Lambda(B)}\,dx$$

$$= \int_B p(x) f(x)\,dx \quad \Rightarrow \quad P_\Phi^T \overset{d}{=} \mathrm{Poi}\,(p(\cdot) f(\cdot)) \text{ point process.}$$

CONSTRUCTION: P = Poisson $(f(\cdot))$ on a bounded set B.

Choose $N_B \overset{d}{=} \mathrm{Poi}(\Lambda(B))$ $X_1, \dots, X_{N_B}$ i.i.d $\frac{f(\cdot)}{\Lambda(B)}$

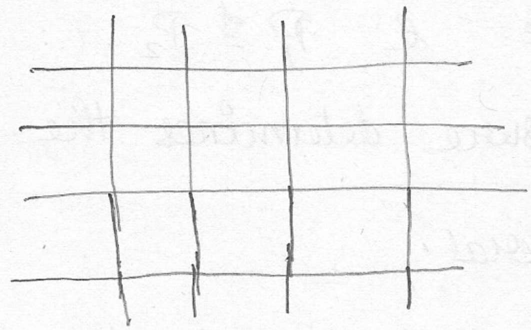
$$\{X_1, \dots, X_{N_B}\} \overset{d}{=} P \cap B.$$

Suppose $B_1 \cap B_2 = \phi$, B_1, B_2 bounded.

Φ_{B_1} $P \cap B_1$ & $P \cap B_2$ are independent.

So construct independently & super-impose.

For general $f(\cdot)$ on $\mathbb{R}^d$



Construct Poisson process on each cube & super-impose. the point process.

COUPLING: Suppose $g(x) = p(x) f(x)$ $\quad p(x) \in [0,1] \ \forall x \in \mathbb{R}^d$

Is $\mathcal{P}_g \subseteq \mathcal{P}_f$ a.s. ? i.e. a thinning.

Clearly $\mathcal{P}^T \overset{d}{=} \mathcal{P}_g$ by uniqueness. So we can choose this as $\mathcal{P}_g$.

CHARACTERIZATIONS:

General theorem on simple point processes:

THM

$$P(\Phi_1(B) = 0) = P(\Phi_2(B) = 0) \quad \forall B \ \ldots$$

$$\Rightarrow \Phi_1 \overset{d}{=} \Phi_2.$$

COR: $P(\Phi(B) = 0) = e^{-\Lambda(B)} \Rightarrow \Phi \overset{d}{=} \mathcal{P}_f$

THM: Φ - simple p.p. & $\forall n$, $B_1, \ldots, B_n$ disjoint bounded sets we have that $\Phi(B_1), \ldots, \Phi(B_n)$ are independent. Then $\Phi \overset{d}{=} \mathcal{P}_f$ on some $f(\cdot)$.

Proof: Suppose set A is small.

$$\Lambda(A) = E\,\Phi(A) = \sum_{i \geq 1} i\, P(\Phi(A) = i) \text{ is small.}$$

The dominant term is $P(\Phi(A) = 1)$.

so $P(\Phi(A) = 1) \approx P(\Phi(A) \geq 1) \approx \Lambda(A)$.

Let B be a bounded set. Partition into 'n' disjoint small pieces A_i

$$P(\Phi(B) = 0) = P(\Phi(A_1) = 0, \ldots, \Phi(A_n) = 0)$$

$$(\text{Independence}) = \prod_{i=1}^{n} P(\Phi(A_i) = 0)$$

$$\approx \prod_{i=1}^{n} (1 - \Lambda(A_i)) \approx \prod_{i=1}^{n} e^{-\Lambda(A_i)}$$

$$= e^{-\Lambda(B)}. \quad \text{(Apply prev. thm)}. \quad ∎$$

$\boxed{\text{ISOTROPY:}}$ ~~$\mathbb{T}$~~ rotations $\mathcal{T}$ = Class of transformations.

If $\Phi \overset{d}{=} T\Phi \;\; \forall T \in \mathcal{T}$, then Φ is $\mathcal{T}$-invariant.

Eg. 1: $\mathcal{T} = \{\text{Translations}\}$ - Φ is Stationary

~~Eg. 2~~ If $\mathcal{P}_f$ is stationary $\Rightarrow \int_B f(x+c)\,dx = \int_B f(x)\,dx$

$\forall\, c \in \mathbb{R}^d$ & $B \subseteq \mathbb{R}^d$. $\Rightarrow f(x+c) = f(x) \;\; \forall\, x \in \mathbb{R}^d, c \in \mathbb{R}^d$

$\Rightarrow f(c) = f(0) \;\; \forall\, c \in \mathbb{R}^d$ i.e., $f \equiv$ constant.

Stationary Poisson point process $\Rightarrow \mathcal{P}_\lambda$ i.e., $f \equiv \lambda$.

Eg. 2 $\mathcal{T} = \{\text{Rotations}\}$ Φ is Isotropic.

$\mathcal{P}_f$ is Isotropic $\Rightarrow f(x) = g(\|x\|)$ for some g.

The Stationary Poisson point process is also ISOTROPIC.

EXTENDED POINT PROCESS.

$$W_n = [-n, n]^d$$

$\mathcal{P}_\lambda$ – Stationary Poisson point process.

$$\mathcal{P}_n := (2n)^{-1} \mathcal{P}_{\lambda,n} = \{ (2n)^{-1} X_i : X_i \in \mathcal{P}_\lambda \cap W_n \}$$

$$\Rightarrow \quad \mathcal{P}_n \subseteq \left[\frac{-1}{2}, \frac{1}{2} \right]^d = U$$

$$\mathcal{P}_n(B) = \mathcal{P}_{\lambda,n}((2n) \cdot B) \overset{d}{=} \mathrm{Poi}(\lambda (2n)^d |B|)$$

$$\Rightarrow \quad \mathcal{P}_n \overset{d}{=} \mathcal{P}_{\lambda(2n)^d f(\cdot)} \quad \text{where } f(\cdot) = \mathbb{1}_U(\cdot).$$

~~$\forall\, X, Y \in \mathcal{P}_{\lambda,n}$ ... $|X - Y| \le$~~

NOTE: ~~Properties of~~ Performance measures of networks on $\mathcal{P}_n$ can be translated to that of $\mathcal{P}_{\lambda,n}$.

PRODUCTS: $\Phi \subseteq \mathbb{R}^d$. $\Phi^{(k)} = \{(x_1, \dots, x_k) : x_i \in \Phi, x_i \neq x_j \ \forall i \neq j\}$
$\subseteq (\mathbb{R}^d)^k$.

Φ is simple $\Rightarrow$ $\Phi^{(k)}$ is simple.

If $B \subseteq (\mathbb{R}^d)^k$ is bounded then $B \subseteq B_1 \times \dots \times B_k$, each bdd in $\mathbb{R}^d$

so $\Phi^{(k)}(B) \leq \prod_{i=1}^{k} \Phi(B_i) < \infty$ a.s.

$\Rightarrow$ $\Phi^{(k)}$ is a point process on $(\mathbb{R}^d)^k$.

Intensity measure: $\alpha^{(k)}(B_1 \times \dots \times B_k) \overset{\text{Always true.}}{=} \mathbb{E}[\Phi^{(k)}(B_1 \times \dots \times B_k)]$

k^{th} factorial moment measure of Φ. $\overset{\text{disjoint}}{\equiv} \mathbb{E}\left[\prod_{i=1}^{k} \Phi(B_i)\right]$

$B_1, \dots, B_k$ - disjoint. bdd sets.

Like before, possible to ask is there an integral representation of $\alpha^{(k)}(\cdot)$ i.e.,

$$\alpha^{(k)}(B) = \int_B e^{(k)}(x)\, dx.$$

Again enough if it holds on $B_1, \dots, B_k$ disjoint bdd sets

$$\alpha^k(B_1 \times \dots \times B_k) = \int_{B_1 \times \dots \times B_k} e^{(k)}(x_1, \dots, x_k)\, dx_1 \dots dx_k$$

If it exists $e^{(k)}(\cdot)$ - k^{th} correlation functions.

$$\alpha^{(k)}(B \times \dots \times B) = \mathbb{E}[\Phi(B)(\Phi(B)-1) \dots (\Phi(B)-k+1)]$$

Hence Factorial moment measure.

Play the role of "moments" for point processes.

$\rho^{(k)}(x_1, \dots, x_k) \approx \text{Prob}(x_1, \dots, x_k \text{ are points of } \Phi)$.

i.e. a.e. $\rho^{(k)}(x_1, \dots, x_k) = \lim_{\varepsilon \to 0} \dfrac{P(\Phi(B_\varepsilon(x_i)) \geq 1 \ \forall\, i \leq k)}{|B_\varepsilon|^k}$

$\boxed{\text{SYMMETRIC}}$

Do joint densities determine the point process?

i.e. let $\forall k$ & $x_1, \dots, x_k \in \mathbb{R}^d$

$$\rho_1^{(k)}(x_1, \dots, x_k) = \rho_2^{(k)}(x_1, \dots, x_k).$$

Then is $\Phi_1 \stackrel{d}{=} \Phi_2$?

i.e., $(\Phi_1(B_1), \dots, \Phi_1(B_k)) \stackrel{d}{=} (\Phi_2(B_1), \dots, \Phi_2(B_k))$

$\forall B_1, \dots, B_k$ compact.

Suppose

$$E\left[\exp\left\{\sum_{i=1}^{n} s_i \Phi_j(B_i)\right\}\right] < \infty \quad (*)$$

then using e^x series expansion of e^x, one can express $E[\exp\{\dots\}]$ in terms of correlation functions & then the equality in distribution follows.

To guarantee $(*)$ – $\forall$ compact B, $\exists\, c > 0$

$$P[\Phi(B) \geq k] \leq e^{-ck} \quad \forall\, k \geq 1.$$

(In particular if $E\, e^{c\Phi(B)} < \infty$ for some $c > 0$).

$\rho^{(2)}(x_1, x_2)$ – Pair correlation function.

$\rho^{(2)}(x_1, x_2) < \rho^{(1)}(x_1)\rho^{(2)}(x_2)$ (Negative correlation)

$>$ (positive correlation).

~~FUNCTIONALS~~:

Eg: $\mathcal{P}_f$. $\quad E\left[\prod_{i=1}^{k} \mathcal{P}_f(B_i)\right] = \prod_{i=1}^{k} E\left[\mathcal{P}_f(B_i)\right]$

$B_1, \ldots, B_k$ bdd disjoint

$$= \int_{B_1} \cdots \int_{B_k} \prod_{i=1}^{k} f(x_i)\, dx_1 \cdots dx_k$$

$$\Rightarrow \rho^{(k)}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f(x_i).$$

$\rightarrow$ Stationarity $\Rightarrow$ $\rho^{(k)}(x_1+y, \ldots, x_k+y) = \rho^{(k)}(x_1, \ldots, x_k)$. $\forall\, y \in \mathbb{R}^d$.

FUNCTIONALS:

Not truly but we've expressions / Idea how to work with $\Phi(B)$ ~~or~~ $(\Phi(B_1), \ldots, \Phi(B_k))$

But what about other functionals / performance measures?

Eg 1: ~~CAMP~~ $H(\Phi) = \sum_{x \in \Phi} h(x)$? $\quad h(x) = \mathbb{1}[x \in B] \quad H(\Phi) = \Phi(B)$.

~~Meo~~ $E\,\Phi(B)$'s $\longrightarrow$ $\boxed{\text{Measure theory}}$ $\longrightarrow$ $EH(\Phi)$. for reasonable "h"?

Let h be a non-negative "nice" function.

CAMPBELL'S THEOREM (MECKE). $\quad EH(\Phi) = \int_{\mathbb{R}^d} h(x)\, \rho(x)\, dx$ $\quad$ Makes sense if both are finite

Eg 2: Observer at origin: $h(x)$ - signal power ~~at~~ received at origin from transmitter at x.

$$H(\Phi) = \sum_{x \in \Phi} h(x) \text{ - total signal received at } 0.$$

$$EH(\Phi) = \int h(x)\, \rho(x)\, dx.$$

Suppose Φ - stationary $\Rightarrow$ $\rho(x) \equiv \lambda = \mathbb{E}[\Phi[U]]$, $U = [-\frac{1}{2}, \frac{1}{2}]^d$

$$\Rightarrow \quad \mathbb{E}[H(\Phi)] = \lambda \int_{\mathbb{R}^d} h(x)\, dx.$$

$\boxed{\text{MEASURE THEORY TRICK:}}$ $h: \mathbb{R}^d \to \mathbb{R}$ i.e., not non-negative.

but $\int_{\mathbb{R}^d} |h(x)|\, dx < \infty$ then Campbell's Theorem still holds.

Work with $h_+(x) = h(x)\, 1[h \geq 0]$ & $h_-(x) = -h(x)\, 1[h < 0]$

& $h = h_+ - h_-$.

REFINED CAMPBELL'S THEOREM:

$\Phi^{(k)}$ — Also point process, Intensity - correlation functions.

$h: (\mathbb{R}^d)^k \longrightarrow [0, \infty)$ -

$$H(\Phi) = \sum_{X_1, \dots, X_k \in \Phi}^{\neq} h(X_1, \dots, X_k) = \sum_{X \in \Phi^{(k)}} h(X)$$

$$\mathbb{E} H(\Phi) = \int_{(\mathbb{R}^d)^k} h(x_1, \dots, x_k)\, \rho^{(k)}(x_1, \dots, x_k)\, dx_1 \dots dx_k.$$

REMARKS:

1). Stationarity not enough to inform about $\rho^{(k)}(\cdot, \cdot)$'s.

Eg: $X \sim \text{unif}\, [-\frac{1}{2}, \frac{1}{2}]^d$ $\quad \Phi = X + \mathbb{Z}^d$.

Φ is stationary. $(X_1, X_2) \in \Phi \Rightarrow X_1 - X_2 \in \mathbb{Z}^d$.

2). If h is symmetric, often $H(\Phi) = \frac{1}{k!} \sum_{X_1, \dots, X_k}^{\neq} h(X_1, \dots, X_k)$.

3) - CAMPBELL'S THEOREM can be used to define correlation functions.

Eg: $h(x,y) = \frac{1}{2} 1[|x-y| \leq r]$. $H(\Phi) = \#$ pairs $(x,y) \in \Phi^{(2)} \ni |x-y| \leq r$.

Alternatively define graph G

$G(\Phi, r)$ - Vertices - Φ ; Edges - $(x,y) \ni 0 < |x-y| \leq r$.

Then $\cancel{\#}\ H(\Phi) = \#$ edges in $G(\Phi, r)$.

$$\mathbb{E}H(\Phi) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} 1[|x-y| \leq r]\, \rho^{(2)}(x,y)\, dx\, dy$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} dx \int_{B_r(x)} \rho^{(2)}(x,y)\, dy .$$

Eg: $h(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)\, 1[\{x_1, \ldots, x_k\} \subseteq W]$

$$\mathbb{E}H(\Phi) = \int_{W^k} g(x_1, \ldots, x_k)\, \rho^{(k)}(x_1, \ldots, x_k)\, dx_1 \cdots dx_k .$$

WHAT MORE ?

$I(r) = \#$ Isolated nodes in $G(\Phi, r)$

$$= \sum_{x \in \Phi} 1[\Phi(B_r(x)) = 1] = \sum_{x \in \Phi} h(x, \Phi - \{x\})$$

POISSON CASE :

$$\mathbb{E}\Big[\sum_{x \in \Phi} h(x, \Phi - \{x\})\Big] = \int_{\mathbb{R}^d} \mathbb{E}[h(x, \Phi)]\, f(x)\, dx .$$

$$\mathbb{E}[I(r)] = \int_{\mathbb{R}^d} \mathbb{P}[\Phi(B_r(x) - \{x\}) = 0]\, f(x)\, dx$$

$$= \int e^{-\int_{B_r(x)} f(y)\, dy} f(x)\, dx .$$

i.e., $H(\Phi) = \sum_{x_1,\dots,x_k} h(x_1,\dots,x_k, \Phi - \{x_1,\dots,x_k\})$

$$EH(\Phi) = \int_{(\mathbb{R}^d)^k} \mathbb{E}\left[h(x_1,\dots,x_k,\Phi)\right] \prod_{i=1}^{k} f(x_i)\, dx_1 \cdots dx_k.$$

GENERAL POINT PROCESS:

Need to know Φ if $x \in \Phi$? Distribution ?

Stationarity $\Rightarrow$ $P(x \in \Phi) = 0$ i.e., conditioning on a null event.

NAIVE APPROACH: Let $\mathbb{P}_x^!(\cdot)$, $\mathbb{E}_x^!(\cdot)$ denote probabilities, expectations w.r.t. conditioning $x \in \Phi - \{x\}$

$$\mathbb{E}_x^!(\Phi(B)) = \lim_{\varepsilon \to 0} \frac{\mathbb{E}\left[\Phi(B)\, \mathbb{1}[\Phi(B_\varepsilon(x)) \geq 1]\right]}{P[\Phi(B_\varepsilon(x)) \geq 1]}$$

SLIVNIYAK'S THEOREM: for $\Phi = \mathcal{P}_f$,

$$\mathbb{E}_x^!(\cdot) = \mathbb{E}(\cdot) \quad \& \quad \mathbb{P}_x^!(\cdot) = P(\cdot)$$

i.e., conditioning on "x" changes nothing.

REFINED CAMPBELL THEOREM:

$$\mathbb{E}[H(\Phi)] = \int_{\mathbb{R}^d} \mathbb{E}_x^!\left[h(x,\Phi)\right] \rho(x)\, dx$$

for $\Phi^{(k)}$

$$\mathbb{E}[H(\Phi^{(k)})] = \int_{(\mathbb{R}^d)^k} \mathbb{E}^!_{x_1 \cdots x_k}\left[h(x_1,\dots,x_k,\Phi)\right] \rho^{(k)}(x_1,\dots,x_k)\, dx_1 \cdots dx_k$$

Version.

Many cases very useful to do analytic calculations but much harder work than Poisson.

Explicit formulas - very rare.

REFERENCES:

1.) M. HAENGGI – "Stochastic geometry for wireless networks"

2.) F. BACCELLI & B. BLASZCZYSZYN – " . . "

3.) B. B., E. MERZBACH, V. SCHMIDT – "Factorial moment expansion for . . "