I. Coverage in a Poisson-Boolean Model

• Motivation

• Let Φ denote a homogenous Poisson point process on $\mathbb{R}^d$ of intensity λ.

$\Phi(A) = \#$ of points falling in the set $A \subseteq \mathbb{R}^d$.

i) $\Phi(A) \sim Poi(\lambda|A|)$, where $|A| =$ volume of A.

$$P[\Phi(A)=k] = e^{-\lambda|A|}\frac{(\lambda|A|)^k}{k!}, \quad k=0,1,2,\ldots$$

$$E[\Phi(A)] = \lambda|A|$$

$\lambda =$ expected # of points falling in a unit volume.

ii) If $A, B \subseteq \mathbb{R}^d$ are disjoint, then $\Phi(A)$ & $\Phi(B)$ are independent random variables.

• Let $\Phi = \{X_1, X_2, \ldots\}$

$B(x,r) =$ ball of radius r centered at x.

$|B(x,1)| = \theta$.

Def. The Poisson-Boolean model or the coverage process is defined as

$$\mathcal{C} = \mathcal{C}(\lambda, r) = \bigcup_{i=1}^{\infty} B(X_i, r)$$

Def. The vacancy of a smooth bounded region $R \subseteq \mathbb{R}^d$ is the region within R that is not covered by $\mathcal{C}$, i.e.

$$V(R) = \int_R 1_{\mathcal{C}^c}(x)\,dx$$

$$\left[1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases} \right]$$

• $E[V(R)] = E\left[\int_R 1_{\mathcal{C}^c}(x)\,dx\right]$

$$= \int_R E[1_{\mathcal{C}^c}(x)]\,dx$$

$$= \int_R P[x \notin \mathcal{C}]\,dx$$

$$= \int_R P[\Phi(B(x,r))=0]\,dx$$

$$= \int_R e^{-\lambda|B(x,r)|}\,dx$$

$$= |R|\, e^{-\lambda\theta r^d}$$

$$\boxed{E[V(R)] = |R|\, e^{-\lambda\theta r^d}}$$

Remark: So if we want not more than α proportion of the region R to be vacant (not covered) on average, then we must have

$$\boxed{r^d \geq \frac{-\log(\alpha)}{\lambda\theta}} \quad \text{or} \quad \boxed{\lambda \geq \frac{-\log\alpha}{r^d\theta}}$$

$$\bullet\ E[V(R)^2] = E\left(\int_R 1_{C^c}(x)\,dx\right)^2$$

$$= E\left[\int_R\int_R 1_{C^c}(x)\,1_{C^c}(y)\,dx\,dy\right]$$

$$= \int_R\int_R P[\Phi(B(x,r)\cup B(y,r))=0]\,dx\,dy$$

$$= \int_R\int_R e^{-\lambda|B(x,r)\cup B(y,r)|}\,dx\,dy$$

$$= e^{-2\lambda\theta r^d}\int_R\int_R e^{-\lambda|B(x,r)\cap B(y,r)|}\,dx\,dy.$$

$$Var(V(R)) = e^{-2\lambda\theta r^d}\int_{R\times R}\left(e^{\lambda|B(x-y,2r)\cap B|}-1\right)dx\,dy.$$

Theoretical Results:-

1) Strong Law: If $\lambda\to\infty$ such that $\lambda\theta r^d \to \rho > 0$, then
$V(R) \to |R|e^{-\rho}$ almost surely

2) $\lambda\, Var(V(R)) \to \sigma^2$ as $\lambda\to\infty$, $\lambda\theta r^d \to \rho > 0$ &

CLT $\sqrt{\lambda}\,[V(R) - E[V(R)]] \xrightarrow{d} N(0,\sigma^2)$.

— x —

Extensions:-

• Random sets

1) Cox process

n – large # of sensors
each sensor is on or off w.p. p.
# of sensors on $\sim Poi(np)$

2) Poisson cluster process

3) k-coverage. 4) On-off process.

Complete Coverage

Q. How should r scale with λ as $\lambda\to\infty$ so that R is completely covered with probability approaching one?

Thm (Hall) Let $d=2$. For each $\lambda>0$ & $0<r\le\frac{1}{2}$ we have

$$\frac{1}{2}\min\{1,(1+\theta r^2\lambda^2)e^{-\theta r^2\lambda}\} < P[V>0] < 3\min\{1,(1+\theta r^2\lambda^2)e^{-\theta r^2\lambda}\}.$$

$\Rightarrow$ Take

$$\theta r^2 = \frac{\log\lambda + \log\log\lambda + c_\lambda}{\lambda}$$

Then $P(V>0)\to 0$ as $\lambda\to\infty$ iff $c_\lambda\to\infty$.

II. PERCOLATION

- Percolation refers to the existence of an infinite/giant component in a graph.

: Phase Transition : sudden emergence of a phenomenon

1. Bond percolation in $\mathbb{Z}^2$ (the random grid)

Graph G_p with vertex set $V = \mathbb{Z}^2$ & edge set

$$\mathcal{E} := \{ \langle x,y \rangle : x, y \in \mathbb{Z}^2, |x-y| = 1 \}$$

Each edge is open w.p. p & closed w.p. $1-p$ independent of other edges. | $U_{xy} \sim U(0,1)$, $P(U_{xy} < p) = p$

This yields the random graph G_p

$$\mathcal{E}_p = \{ \langle x,y \rangle \in \mathcal{E} : \langle x,y \rangle \text{ is open} \}.$$

$$G_p = (\mathbb{Z}^2, \mathcal{E}_p)$$

Def. A ~~conne~~ path is said to exist from $u \in \mathbb{Z}^2$ to $v \in \mathbb{Z}^2$ if $\exists$ a seq. of vertices $v_0, \ldots, v_n$ s.t. $u = v_0$, $v_n = v$ & $\langle v_{i-1}, v_i \rangle \in \mathcal{E}_p$, $\forall i = 1, \ldots, n$. $(u \to v)$

Def. A connected component is a maximal set of vertices s.t. for any two vertices in the set there is a path from one to the other.

Def. The network G_p is said to percolate if it contains an infinite connected component.

$\psi(p) = \mathbb{P}(\exists$ an infinite connected component in $G_p)$

$= \mathbb{P}(G_p$ percolates$)$.

Def. A phase transition is said to occur at a critical point $p_c \in (0,1)$ if $\psi(p) = 0$ $\forall p < p_c$ & $\psi(p) = 1$ $\forall p > p_c$.

Def. $C(x)$ = connected component containing $x \in \mathbb{Z}^2$.

$C = C(0)$.

$|C(x)|$ = cardinality of $C(x)$.

$\theta(p) = \mathbb{P}_p(|C(x)| = \infty)$

- percolation probability.

Thm. If $0 < p_1 < p_2 < 1$ then $\theta(p_1) \le \theta(p_2)$.

Thm. (Kesten 1980) $\cdot$ $p_c = \frac{1}{2}$ | Easy to show $\frac{1}{3} < p_c < \frac{2}{3}$

: $\theta(p_c) = 0$

- At most one infinite connected comp

Cor $\psi(p) = 0$ for $p \le \frac{1}{2}$ &

$\psi(p) = 1$ for $p > \frac{1}{2}$.

Pf. $\cdot$ $\psi(p) = \mathbb{P}_p\left(\bigcup_x \{|C(x)| = \infty\} \right)$

$\le \sum_x \theta(p) = 0$. if $p \le p_c$.

- percolation of G_p does not depend on the state of any finite collection of edges.

Kolmogorov's 0-1 law $\Rightarrow \psi(p) = 0$ or 1

$\psi(p) \ge \theta(p) > 0$ if $p > p_c$.

$\Rightarrow \psi(p) = 1$ if $p > p_c$.

Implication for finite Graphs.

Let $G_{n,p}$ be G_p restricted to edges in $[0, \sqrt{n}]^2$. Then for large n, if $p < p_c$ then components are of size $O(\log n)$. If $p > p_c$ then largest component is $O(n)$ & second largest is $O(\log n)$. whp

III. The Random Connection Model

• Motivation

Φ_λ be a homogeneous Poisson point process on $\mathbb{R}^2$ of intensity λ.

$g : \mathbb{R}^2 \to [0,1]$ $g(x)$ depends only of $|x|$ & is non-increasing

$X, Y \in \Phi_\lambda$ are connected by an edge w.p. $g(|X-Y|)$

Assumption: $0 < \int_{\mathbb{R}^2} g(x)\,dx < \infty$

Let $\mathbb{P}^x$ denote the Palm measure, i.e. the distribution of Φ_λ conditioned to have a point at $\{x\}$.

• Distribution of Φ_λ under $\mathbb{P}^x$ is same as that of $\Phi \cup \{x\}$ under $\mathbb{P}$.

$\tilde{\Phi}$ be the points of Φ_λ connected by an edge to $\{x\}$ under $\mathbb{P}^x$.

• $\tilde{\Phi}$ is a non-homogeneous Poisson point process of intensity $\lambda g(\cdot)$.

$$E[\tilde{\Phi}(\mathbb{R}^2)] = \lambda \int_{\mathbb{R}^2} g(y)\,dy$$

The reason for the assumption above.

$|C|$ = cardinality of component containing origin O under $\mathbb{P}^o$.

$\theta(\lambda) = \mathbb{P}^o(|C| = \infty)$

Thm:- $\exists$ a $\lambda_c \in (0,\infty) \ni \theta(\lambda) = 0$ for $\lambda < \lambda_c$ & $\theta(\lambda) > 0$ for $\lambda > \lambda_c$.

IV The Boolean Model

$$g(x) = \begin{cases} 1 & \text{if } |x| < 2r \\ 0 & \text{o.w.} \end{cases}$$

Average degree of a node (ξ) $= 4\pi r^2 \lambda$.

Thm: i) Fix $r > 0$. $\exists$ a $\lambda_c > 0 \ni$ "$\lambda_c(r)$" for $\lambda < \lambda_c$, $\theta(\lambda) = 0$ & for $\lambda > \lambda_c$ $\theta(\lambda) > 0$

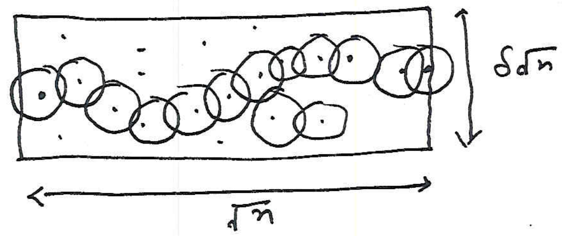
ii) Fix $\lambda > 0$. $\exists$ a $r_c > 0 \ni$ for $r < r_c(\lambda) \ni$ ~~for~~ $\theta(r) = 0$ & for $r > r_c$ $\theta(r) > 0$

iii) $\exists$ a $\xi_c \ni$ for $\xi > \xi_c$, $\theta(\xi) > 0$ & for $\xi < \xi_c$ $\theta(\xi) = 0$.

$$\xi_c = 4\pi r_c^2 \lambda_c^2 \simeq 4.512$$

Thm: Fix $r > 0$ & let $\lambda > \lambda_c$. $R^{\leftrightarrow}_{\delta n}$ denote a left-right crossing of a rectangle of sides $\sqrt{n}, \delta\sqrt{n}$ resp. Then

$$\mathbb{P}(R^{\leftrightarrow}_{\delta n}) \to 1 \quad \text{as } n \to \infty, \forall \delta > 0$$



V Connectivity

Almost Connectivity

Φ - Homogeneous $PPP(1)$ on $\mathbb{R}^2$.

$B_n = [0, \sqrt{n}]^2$

$G_n(r)$ = graph with vertex set $\Phi \cap B_n$ & edges between any two pair of points within distance $2r$.

Remark:- All results stated here hold for $\tilde{G}_n(\tilde{r})$ with vertex set $\frac{1}{\sqrt{n}} \Phi \cap B_1$ & $\tilde{r} = \frac{r}{\sqrt{n}}$.

$N_\infty(B_n)$ = # of points in $\Phi \cap B_n$ that are part of the infinite component in the Boolean model with vertex set Φ & radius r.

Call this graph $G(r)$

Prop. $\theta(r) = \mathbb{P}^0(\text{origin percolates}) = E[N_\infty(B_1)]$

Pf. Apply Campbell-Mecke formula.

Def. For any $\alpha \in (0,1)$, $G_n(r)$ is said to be α-almost connected if it contains a connected component of at least αn vertices.

Thm: Let $r_\alpha := \inf\{r : \theta(r) > \alpha\}$, $\alpha \in (0,1)$. If $r > r_\alpha$ then $G_n(r)$ is α-almost connected a.a.s. & if $r < r_\alpha$ it is not.

Full Connectivity

$\Phi \sim HPPP(1)$ on $\mathbb{R}^2$

$B_n = [0, \sqrt{n}]^2$ $\Phi_n = \Phi \cap B_n$

To avoid edge effects, we will take metric on B_n to be the toroidal metric:

$$\tilde{d}_n(x,y) = \inf_{z \in \sqrt{n}\mathbb{Z}} d(x, y+z)$$

$G_n(r)$ = graph with vertex set Φ_n & edges between any two pairs of ~~points~~ vertices at dist. $\tilde{d} \le 2r$.

W_n = # of isolated nodes in $G_n(r_n)$

$$E[W_n] = n e^{-\pi(2r_n)^2}$$

So if $\pi(2r_n)^2 = \log n + \alpha_n$ & $\alpha_n \to \alpha$ then

$$E[W_n] \to e^{-\alpha}$$

Remark:- 1. For dense network, we should take $\pi(2\tilde{r}_n)^2 = \frac{\log n + \alpha_n}{n}$

2. $E[W_n] \to e^{-\alpha} \Rightarrow$ chance of any node being isolated is tending to 0. Dependence is local.

Thm: If $\pi(2r_n)^2 = \log n + \alpha_n$, $\alpha_n \to \alpha$ then $W_n \xrightarrow{d} Po(e^{-\alpha})$.

$$\mathbb{P}[W_n = 0] \to e^{-e^{-\alpha}}, \quad \alpha \in \mathbb{R}.$$

Remark: Thus to eliminate isolated nodes we must have $\alpha_n \to \infty$. Remarkably this suffices to connect the graph!!

Thm: If $\pi(2r_n)^2 = \log n + \alpha_n$, then $G_n(r_n)$ is connected w.h.p. iff $\alpha_n \to \infty$.

Interference Limited Networks

1 Static SINR Graph (without fading)

$\Phi \sim HPPP(\lambda)$ on $\mathbb{R}^2$

$l : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ be such that

i) $l(x,y) = h(|x-y|)$ for some $h : \mathbb{R}_+ \to \mathbb{R}_+$, cont., strictly decreasing on the set where it is >0.

ii) $\int_0^\infty r h(r) dr < \infty$; $h(r) \le 1 \ \forall r \ge 0$.

l: path loss fn.

Let P, T, N be positive parameters & $\gamma \ge 0$.

iii) $h(0) > \frac{TN}{P}$.

$$I_{xy} := \sum_{z \in \Phi \setminus \{x,y\}} P\, l(|x-y|)$$

$$R_{xy} := \frac{P\, l(|x-y|)}{N + \gamma I_{xy}}, \quad x,y \in \Phi.$$

Def. The static SINR graph with vertex set Φ and directed edge set

$E = \{(x,y) : x,y \in \Phi, R_{xy} > T\}$

$|C|$ = # of points of Φ in the cluster (component) containing the origin under $\mathbb{P}^0$.

Graph is said to percolate if $\mathbb{P}^0(|C| = \infty) > 0$.

Note:- i) Length of edges unif. bounded by $h^{-1}\left(\frac{TN}{P}\right)$

ii) $\gamma = 0$ reduces the model to a standard Boolean model.

iii) $\gamma = 0$ & fading we get the random connection model.

Prop 1:- For any $\gamma > 0$ any node in the static SINR graph is connected to at most $1 + \frac{1}{\gamma T}$ neighbours.

Thm 2:- Let λ_c be the critical node density for the graph to percolate when $\gamma = 0$. For any node density $\lambda > \lambda_c$, there exists a $\gamma^*(\lambda) > 0$ such that for any $\gamma < \gamma^*(\lambda)$ the static SINR graph percolates.

Thm:- For $\lambda \to \infty$ we have that $\gamma^*(\lambda) = O\left(\frac{1}{\lambda}\right)$.

So as the intensity increases the interference effect dominates the opposing effect that facilitates formation of edges due to larger availability of nodes.