Determinantal point processes

what they are why they may interest you

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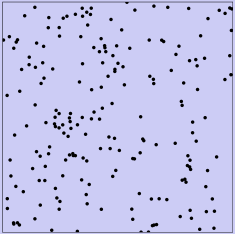
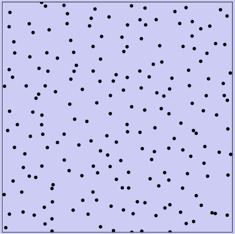
Two pictures

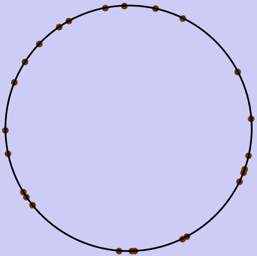


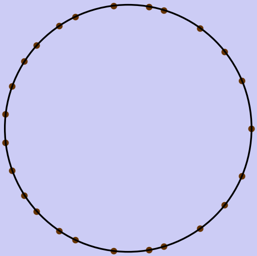


Left: **Poisson process** Right: **Determinantal process**

In both cases the points are randomly distributed and the average number of points per unit area is the same. But one is more evenly spaced out than the other.

Two more pictures





Left: **i.i.d. points** Right: **A determinantal process**

Purpose of the lecture is to ask the question Many models in wireless communication etc., based on Poisson process. Is it worth trying to analyse the same models on a determinantal process?

The setting

Ingredients:

- $\mathcal{E}$ is $\mathbb{R}$ or $\mathbb{C}$ or S^1 or $\mathbb{Z}$ or $\mathbb{R}^d$ .... ("space")
- $K : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$, "kernel function" satisfying
  1. $K(x,y) = \overline{K(y,x)}$. [Symmetry]
  2. $\int_{\mathcal{E}} K(x,y)K(y,z)\,dy = K(x,z)$. [Reproducing property]
  3. $\int_{\mathcal{E}} K(x,x)\,dx = n$, a positive integer. [Rank condition]

Standing remark: If $\mathcal{E} = \mathbb{Z}$ or any finite or countable set, replace integrals by sums.

- Reproducing property: $\sum_{y \in \mathcal{E}} K(x,y)K(y,z) = K(x,z)$.
- Rank condition: $\sum_{x \in \mathcal{E}} K(x,x) = n$.

Determinantal density

The key Lemma: Let $f(x_1, \ldots, x_n) = \frac{1}{n!}\det(K(x_i, x_j))_{i,j \le n}$ for $x_1, \ldots, x_n \in \mathcal{E}$.

1. f is a probability density on $\mathcal{E}^n$.
2. f is symmetric in its arguments.
3. Any k-dimensional marginal of f is given by

$$\int_{\mathcal{E}^{n-k}} f(x_1, \ldots, x_n) dx_{k+1} \ldots dx_n = \frac{(n-k)!}{n!} \det(K(x_i, x_j))_{i,j \le k}.$$

Such an f is called a **determinantal density** [Macchi (1975-76)].

Remarkable point: These multivariate densities can be integrated out to get the marginals explicitly!

Proof of the lemma

There is only one key step: Show that

$$\int_{\mathcal{E}} \det(K(x_i, x_j))_{i,j \le n} \, dx_n = \det(K(x_i, x_j))_{i,j \le n-1}.$$

Step 1: Write

$$\det(K(x_i, x_j))_{i,j \le n} = \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn}(\pi) \prod_{\ell=1}^{n} K(x_\ell, x_{\pi(\ell)}).$$

Step 2: For each $\pi \in \mathcal{S}_n$, compute

$$\int_{\mathcal{E}} \prod_{\ell=1}^{n} K(x_\ell, x_{\pi(\ell)}) \, dx_n.$$

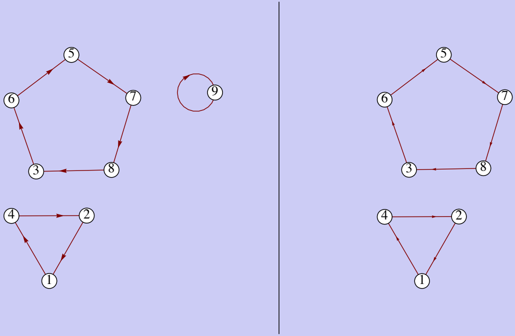
Step 3: Add them all up and get $\det(K(x_i, x_j))_{i,j \le n-1}$.

Proof of the lemma [cont'd]

Want to compute

$$\int_{\mathcal{E}} \prod_{\ell=1}^{n} K(x_\ell, x_{\pi(\ell)})\, dx_n.$$

Case 1: $\pi(n) = n$. Integral gives $\quad +n \cdot \prod_{\ell=1}^{n-1} K(x_\ell, x_{\sigma(\ell)})$.

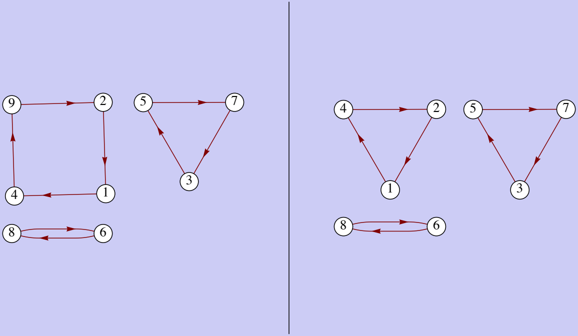


Proof of the lemma [cont'd]

Want to compute

$$\int_{\mathcal{E}} \prod_{\ell=1}^{n} K(x_\ell, x_{\pi(\ell)})\, dx_n.$$

Case 2: $\pi(n) \neq n$. Integral gives $\quad -\prod_{\ell=1}^{n-1} K(x_\ell, x_{\sigma(\ell)})$.



Proof of the lemma [cont'd]

A given $\sigma \in \mathcal{S}_{n-1}$ comes once from case 1 and $n-1$ times from Case 2. Thus,

$$\int_{\mathcal{E}} f(x_1, \ldots, x_n) dx_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_{n-1}} \mathrm{sgn}(\sigma) \prod_{\ell=1}^{n-1} K(x_\ell, x_{\sigma(\ell)})$$
$$= \frac{1}{n!} \det(K(x_i, x_j))_{i,j \leq n-1}.$$

The integration over $x_{n-1}, x_{n-2}, \ldots$ can be continued in exactly the same way.

Determinantal density: summary

Let $K : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C}$ be

1. symmetric: $K(x,y) = \overline{K(y,x)}$,
2. reproducing: $\int_{\mathcal{E}} K(x,y)K(y,z)dy = K(x,z)$ and
3. have rank n: $\int_{\mathcal{E}} K(x,x)dx = n$.

Then $f(x_1,\ldots,x_n) = \frac{1}{n!}\det(K(x_i,x_j))_{i,j\leq n}$ is a symmetric probability density on $\mathcal{E}^n$ whose marginals are of the form

$$\frac{(n-k)!}{n!}\det\left(K(x_i,x_j)\right)_{i,j\leq k}.$$

Remarkable feature: A multivariate density whose marginals can be explicitly found!

Are there such kernels? Yes, many!

Define $K(x,y) = \psi_1(x)\overline{\psi}_1(y) + \ldots + \psi_n(x)\overline{\psi}_n(y)$
where $\psi_1, \ldots, \psi_n$ are orthonormal in $L^2(\mathcal{E})$. That is,

$$\int_{\mathcal{E}} \psi_j(x)\overline{\psi}_k(x)\, dx = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Symmetry: Obvious.

Reproducing property:

$$\int_{\mathcal{E}} K(x,y)K(y,z)dy = \sum_{j,k} \psi_j(x)\overline{\psi}_k(z) \int_{\mathcal{E}} \overline{\psi}_j(y)\psi_k(y)dy$$
$$= \sum_j \psi_j(x)\overline{\psi}_j(z) \;=\; K(x,z).$$

Rank condition:

$$\int_{\mathcal{E}} K(x,x)dx = \sum_{j=1}^{n} \int_{\mathcal{E}} |\psi_j(x)|^2 dx = n.$$

A motivation from quantum physics

Question: Probability densities $f_1, \ldots, f_n$ on $\mathbb{R}$. Simplest density on $\mathbb{R}^n$ made out of these?

Answer: $f_1(x_1)f_2(x_2)\ldots f_n(x_n)$. Independent particles.

Question: *Quantum amplitudes* $\psi_1, \ldots, \psi_n$: complex-valued functions such that $|\psi_k|^2$ is a density. What is the simplest n-particle amplitude made out of these?

$\psi_1(x_1)\ldots\psi_n(x_n)$? Not acceptable as elementary particles are *indistinguishable*.

If we assume that ψ_k are orthogonal to each other, there are two acceptable answers.

- Symmetric amplitude: $\frac{1}{\sqrt{n!}}\sum_{\pi\in\mathcal{S}_n}\prod_{k=1}^{n}\psi_{\pi(k)}(x_k)$
- Anti-symmetric: $\frac{1}{\sqrt{n!}}\sum_{\pi\in\mathcal{S}_n}\operatorname{sgn}(\pi)\prod_{k=1}^{n}\psi_{\pi(k)}(x_k)$.

In either case, $|\text{Amplitude}|^2$ is symmetric.

A motivation from quantum physics [cont'd]

Anti-symmetric case: Write $\frac{1}{n!}\left|\det\left(\psi_j(x_k)\right)_{j,k\leq n}\right|^2$ as

$$\frac{1}{n!}\det\begin{bmatrix}\psi_1(x_1) & \dots & \psi_n(x_1)\\ \vdots & \vdots & \vdots\\ \psi_1(x_n) & \dots & \psi_n(x_n)\end{bmatrix}\begin{bmatrix}\overline{\psi}_1(x_1) & \dots & \overline{\psi}_1(x_n)\\ \vdots & \vdots & \vdots\\ \overline{\psi}_n(x_1) & \dots & \overline{\psi}_n(x_n)\end{bmatrix}$$

$$= \frac{1}{n!}\det\left(K(x_i,x_j)\right)_{i,j\leq n}$$

where $K(x,y)=\sum_{\ell=1}^{n}\psi_\ell(x)\overline{\psi}_\ell(y)$. Determinantal density!!

Example 1: Circular unitary ensemble (CUE)

Let $\mathcal{E} = \mathbb{S}^1$, the unit circle. Let $\psi_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta}$, $0 \leq k \leq n-1$. Then,

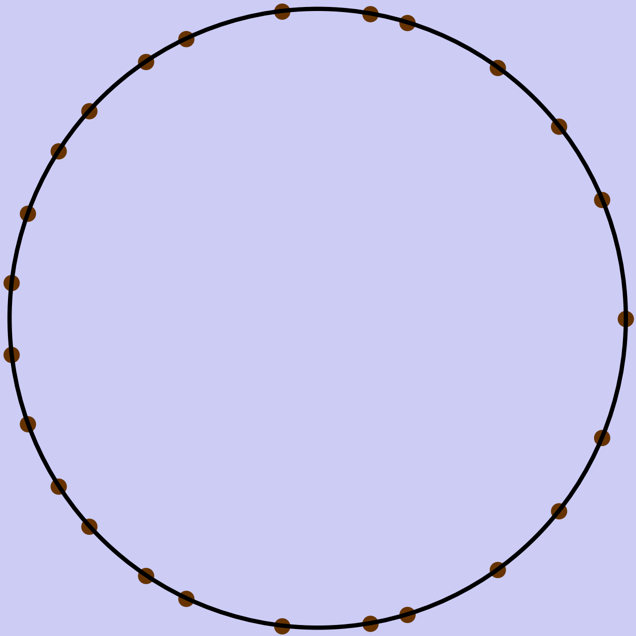
$$K_n(\theta, \phi) = \frac{1}{2\pi} \sum_{k=0}^{n-1} e^{ik\theta} e^{-ik\phi} = \frac{1}{2\pi} \frac{\sin[(n+\frac{1}{2})(\theta-\phi)]}{\sin[\frac{1}{2}(\theta-\phi)]}.$$

The density $\frac{1}{n!} \det(K_n(\theta_i, \theta_j))_{i,j \leq n}$ can also be written as

$$\frac{1}{n!} \left| \det \left(\psi_{j-1}(\theta_k) \right)_{1 \leq j,k \leq n} \right|^2 = \frac{1}{(2\pi)^n n!} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2.$$

Probabilistic origin: Pick a random $n \times n$ unitary matrix according to the *Haar measure*. The eigenvalues have the CUE density (Weyl, Dyson).

Example 1: CUE simulation



Example 2: Ginibre ensemble

Let $\mathcal{E} = \mathbb{C}$, the complex plane. Let $\psi_k(z) = \frac{1}{\sqrt{\pi}\sqrt{k!}} z^k e^{-|z|^2}$.
Then,

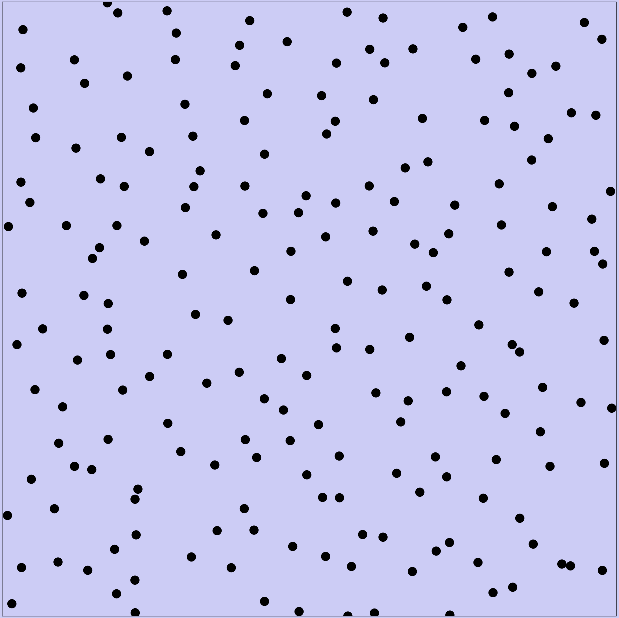
$$K_n(z, w) = \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(z\overline{w})^k}{k!}.$$

The density $\frac{1}{n!} \det(K_n(z_i, z_j))_{i,j \leq n}$ can also be written as

$$\frac{1}{\pi^n n!} \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{k=1}^{n} e^{-|z_k|^2}.$$

Probabilistic origin: Pick a random $n \times n$ matrix whose entries are independent complex Gaussian random variables. The eigenvalues have the Ginibre density (Ginibre-1965).

Example 2: Ginibre simulation



Example 3: Spherical ensemble

Let $\mathcal{E} = \mathbb{C}$ and let $\psi_k(z) = \sqrt{\frac{n}{\pi}\binom{n-1}{k}} \frac{z^k}{(1+|z|^2)^{\frac{1}{2}(n+1)}}$. Then,

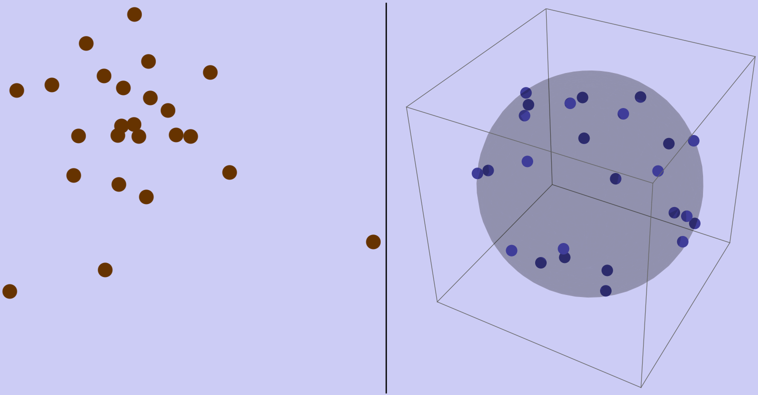
$$K_n(z,w) = \frac{1}{\pi} \frac{(1+z\overline{w})^{n-1}}{(1+|z|^2)^{\frac{n+1}{2}}(1+|w|^2)^{\frac{1}{2}(n+1)}}.$$

The density $\frac{1}{n!}\det(K_n(z_i,z_j))_{i,j\le n}$ can also be written as

$$\frac{1}{\pi^n n!}\prod_{j<k}|z_j - z_k|^2 \cdot \prod_{k=1}^{n} \frac{1}{(1+|z_k|^2)^{n+1}}.$$

Probabilistic origin: Pick two random $n \times n$ matrices A, B whose entries are independent complex Gaussian random variables. The eigenvalues of $A^{-1}B$ have the spherical density (K.-2006).

Example 3: Spherical ensemble simulation



Equivalent: Pick n points on the sphere with density

$$\prod_{j<k} \|P_j - P_k\|^2_{\mathbb{R}^3}.$$

Many examples from random matrix theory

Many random matrices have eigenvalues with determinantal densities. Some example below. All matrices, A, B, A_i are independent and have i.i.d. standard complex Gaussian entries.

- (GUE). $A + A^*$ has n real eigenvalues with density $\prod_{j<k}(x_j - x_k)^2 \cdot e^{-\sum_{j=1}^n x_j^2}$ on $\mathbb{R}^n$. Kernel can be written in terms of Hermite polynomials. (Wigner 1950s). The most studied determinantal density...
- $A_1 A_2 \ldots A_m$. (Akemann and Burda 2012).
- $A_1^{\pm 1} A_2^{\pm 1} \ldots A_m^{\pm 1}$. (K. Adhikari, N. Reddy, T. Reddy, K. Saha - 2014).
- $m \times m$ sub-matrix of a random unitary matrix. (Zyczkowski and Sommers - 2000)

A standard class of examples in the line and plane

Let $\mathcal{E}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $w : \mathcal{E} \mapsto \mathbb{R}_+$ be any weight function. Then, the density

$$C_n \cdot \prod_{k=1}^{n} w(z_k) \cdot \prod_{j<k} |z_j - z_k|^2$$

is a determinantal density. The kernel is

$$K_n(z, w) = \sum_{j=0}^{n-1} \psi_j(z) \overline{\psi_j}(w)$$

where $\psi_0, \ldots, \psi_{n-1}$ are got by applying Gram-Schmidt to $\sqrt{w(z)}, z\sqrt{w(z)}, \ldots, z^{n-1}\sqrt{w(z)}$. [Want to see why?]

Remark: The density $C_n \cdot \prod_{k=1}^{n} w(z_k) \cdot \prod_{j<k} |z_j - z_k|^{2.1}$ appears similar to the previous one qualitatively. But no explicit formulas for the marginal densities!

Example 4: Uniform spanning tree

Let $G=(V,E)$ be a finite graph. Orient each edge in one of two possible ways. Regard each edge as a unit resistor. Define the *transfer-current* matrix

$$K(e,f) = \text{current flowing through } f \text{ when unit current flows from tail of } e \text{ to head of } e.$$

K turns out to have symmetry, reproducing property and rank $|V|-1$. Hence it defines a determinantal pmf on $E^{|V|-1}$.

Probabilistic origin: Among all spanning trees of G, pick one uniformly at random. Regard T as a random subset of E containing $|V|-1$ edges. Then,

$$\mathbb{P}\{T=\{e_1,\ldots,e_{|V|-1}\}\} = \det(K(e_i,e_j))_{i,j\leq|V|-1}.$$

[Burton and Pemantle - 1993].

General properties?

Although the definition looks esoteric, we have shown a good number of natural examples of determinantal densities. We claimed that they are also amenable to analytical and computational study. To justify we show:

1. An algorithm to simulate from a general determinantal density.
2. That the distribution of the number of points that fall in a given region may be exactly computed. In particular, the mean and variance.
3. Asymptotics of determinantal densities reduce to asymptotics of the kernels.

How to sample from a determinantal density?

The following algorithm was introduced in HKPV (2006). Recall that $K : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ is symmetric, reproducing and has rank n.

1. $i = 1$.
2. $\frac{1}{n} K(x,x)$ is a density on $\mathcal{E}$. Sample a point U from this density. Set $X_i = U$.
3. Reset the kernel to

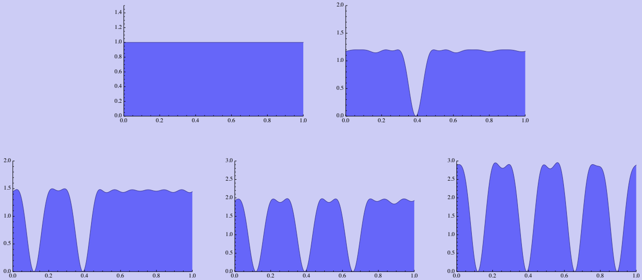
$$K(x,y) \leftarrow K(x,y) - \frac{K(x,U)K(U,y)}{K(U,U)}.$$

4. $i = i + 1$ and $n = n - 1$. If $K = 0$, stop; Else go to Step 2.

Proof: Check that the new kernel in Step 3 is symmetric, reproducing and has one rank less.

Algorithm: illustration

Let $\mathcal{E} = \mathbb{S}^1$ and $K(\theta, \phi) = \frac{1}{2\pi} \frac{\sin[(6+\frac{1}{2})(\theta-\phi)]}{\sin[\frac{1}{2}(\theta-\phi)]}$ (CUE with $n = 6$).



After two more steps, the kernel becomes zero identically and the process stops.

Distribution of points

Let $\frac{1}{n!}\det(K(x_i, x_j))_{i,j\leq n}$ be a determinantal density on $\mathcal{E}^n$. Let $(X_1, \ldots, X_n)$ be a sample from this density. What we care about is the set $\mathcal{X} := \{X_1, \ldots, X_n\}$, not the order. $\mathcal{X}$ is a *point process* with a total of n points, almost surely.

Theorem

For $A \subseteq \mathcal{E}$, let $\mathcal{X}(A)$ be the number of points of $\mathcal{X}$ that fall in A.

1. $\mathbb{E}[\mathcal{X}(A)] = \int_A K(x, x)\, dx$ *and* $Var(\mathcal{X}(A)) = \int_A \int_{A^c} |K(x, y)|^2\, dx\, dy$.
2. $\mathcal{X}(A) \overset{d}{=} \xi_1 + \xi_2 + \ldots$, *where* ξ_k *are independent* $Ber(\lambda_k)$ *random variables where the parameters* $\lambda_1, \lambda_2, \ldots$ *are determined by* K *and* A.

The last statement is implicit in a paper of Shirai and Takahashi (2003) and made explicit and used in HKPV (2006).

Distribution of points [cont'd]

Expected number of points: If $A \subseteq \mathcal{E}$, then

$$\begin{aligned}\mathbb{E}[\mathcal{X}(A)] &= \sum_{k=1}^{n} \mathbb{P}\{X_k \in A\} \\ &= n \int_A \frac{1}{n} K(x,x)\, dx \\ &= \int_A K(x,x)\, dx.\end{aligned}$$

Distribution of points [cont'd]

Second moment:

$$\mathbb{E}[\mathcal{X}(A)(\mathcal{X}(A)-1)] = 2\sum_{1\le j<k\le n} \mathbb{P}\{X_j \in A, X_k \in A\}$$

$$= 2\binom{n}{2}\mathbb{P}\{X_1 \in A, X_2 \in A\}$$

$$= n(n-1)\int_{A\times A} \frac{(n-2)!}{n!}(K(x,x)K(y,y) - K(x,y)K(y,x))dydx$$

$$= \left(\int_A K(x,x)dx\right)^2 - \int_A\int_A K(x,y)K(y,x)\,dy\,dx.$$

Then, $\mathrm{Var}(\mathcal{X}(A)) = \mathbb{E}[\mathcal{X}(A)(\mathcal{X}(A)-1)] + \mathbb{E}[\mathcal{X}(A)] - \mathbb{E}[\mathcal{X}(A)]^2$ which is equal to

$$= -\int_A\int_A K(x,y)K(y,x)\,dy\,dx + \int_A K(x,x)dx$$

$$= \int_A\int_{A^c} |K(x,y)|^2\,dy\,dx.$$

Distribution of points [cont'd]

Full distribution: One can compute all the moments of $\mathcal{X}(A)$ in terms of the kernel K. Then one can check that it has the same distribution as $\xi_1 + \xi_2 + \ldots$ where $\xi_j \sim \text{Ber}(\lambda_j)$ are independent.

The parameters λ_j: Consider the operator $T_A : L^2(A) \to L^2(A)$ defined as

$$(T_A f)(x) = \int_A K(x,y) f(y)\, dy.$$

T_A is a *compact operator* and has eigenvalues $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$. These are the parameters of the Bernoullis.

It is also possible to write the moment generating function of the joint distribution of $\mathcal{X}(A), \mathcal{X}(B), \mathcal{X}(C), \ldots$, but the formulas are a tad complicated looking, so we do not present them.

Infinite determinantal point processes?

Is it possible to define an infinite collection of points that are determinantal in some sense?

As stated, our definition involves densities, hence does not make sense.

But...if $(X_1, \ldots, X_n)$ has determinantal density, by the symmetry of the density, there is no meaning to the first point, second point, etc. Hence, better to not think of the random vector $(X_1, \ldots, X_n)$ but the random set $\{X_1, \ldots, X_n\}$ or equivalently the random measure $\sum_{k=1}^{n} \delta_{X_k}$. The right notion is

Point process: A random (finite or infinite, but locally finite) set of points in $\mathcal{E}$.

How to describe a random collection of points?

Let $\mathcal{X}$ be a random countable subset of $\mathcal{E}$ (we say that $\mathcal{X}$ is a *point process*) with the property that there are only finitely many points in any bounded set.

Joint intensities: Suppose that

$$p_k(x_1, \ldots, x_k) = \lim_{\epsilon \to 0} \frac{\mathbb{P}\{\mathcal{X} \text{ has one point in each of } B(x_j, \epsilon),\ j \leq n\}}{\prod_{k=1}^{n} \text{Vol}(B(x_j, \epsilon))}$$

exists for all $k \geq 1$ and all $x_1, \ldots, x_n \in \mathcal{E}$.

Remark: If $\mathcal{E}$ is $\mathbb{Z}$ or $\mathbb{Z}^d$ etc., we could simply set

$$p_k(x_1, \ldots, x_k) = \mathbb{P}\{\mathcal{X} \text{ contains } x_1, \ldots, x_k\}.$$

If $\mathcal{E} = \mathbb{R}^d$, the probability on the right is zero, hence the more indirect definition.

Joint intensities determine the point process

If $\mathcal{X}$ is a point process with joint intensities p_k, $k \geq 1$, then

$$\mathbb{E}[\mathcal{X}(A)] = \int_A p_1(x)dx,$$

$$\mathbb{E}\left[\prod_{j=1}^{k}(\mathcal{X}(A) - j + 1)\right] = \int_{A^k} p_k(x_1, \ldots, x_k)\, dx_1 \ldots dx_k,$$

$$\mathbb{E}[\mathcal{X}(A)\mathcal{X}(B)] = \int_A \int_B p_2(x, y)\, dxdy \quad \text{if } A \cap B = \emptyset$$

etc. Thus, joint intensities determine all the moments of $\mathcal{X}(A)$ and joint moments of $\mathcal{X}(A)$ and $\mathcal{X}(B)$, etc.

Fact: Assume that $\mathcal{X}(A)$ has exponential decay of tails for any $A \subseteq \mathcal{E}$. Then the joint intensities of $\mathcal{X}$ determine the entire distribution of $\mathcal{X}$.

Example: Poisson process

Let $\lambda : \mathcal{E} \mapsto \mathbb{R}_+$ be any locally integrable function. Then, the Poisson process with intensity function λ is the point process whose joint intensities are

$$p_k(x_1, \ldots, x_k) = \prod_{j=1}^{k} \lambda(x_j).$$

From this one gets independence of $\mathcal{X}(A)$ and $\mathcal{X}(B)$ if $A \cap B = \emptyset$ etc.

Determinantal point process

Let $\mathcal{X}$ be a (simple) point process on $\mathcal{E}$ whose joint intensities are of the form:

$$p_k(x_1, \ldots, x_k) = \det\left(K(x_i, x_j)\right)_{i,j \leq k}$$

for all k and all $x_1, \ldots, x_k$, for some particular kernel $K : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$.
Then, $\mathcal{X}$ is said to be a **determinantal point process** with kernel K [Macchi - 1975-76].

Example: Determinantal densities

Let K be a rank n, symmetric, reproducing kernel on $\mathcal{E}$. Then we defined the determinantal density $\frac{1}{n!}\det(K(x_i, x_j))_{i,j\leq n}$.

If $(X_1, \ldots, X_n)$ is a sample from this density, regard it now as a point process $\mathcal{X} := \{X_1, \ldots, X_n\}$. Then $\mathcal{X}$ is a determinantal point process with kernel K.

Which kernels define determinantal point processes?

Theorem: [Macchi, Soshnikov]. Let $K : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C}$ be

1. Symmetric: $K(x, y) = \overline{K(y, x)}$.
2. Locally of trace class: $\int_A K(x, x)\, dx < \infty$ if A is bounded.

Then, K defines a determinantal point process if and only if the integral operator T_K define by K satisfies $0 \leq T_K \leq I$.

Some examples If $K(x, y) = \sum_{j=0}^{\infty} \psi_j(x)\overline{\psi}_j(y)$ where ψ_j are orthonormal in L^2, then this condition is satisfied.

Example 1: Infinite Ginibre ensemble

Let $\mathcal{E} = \mathbb{C}$ and

$$K(z,w) = e^{z\overline{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2} = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2} \sum_{n=0}^{\infty} \frac{1}{n!} (z\overline{w})^n.$$

The corresponding determinantal point process is called the infinite Ginibre ensemble (limit of the finite Ginibre ensembles). It is a *translation and rotation-invariant* point process in the plane. The intensity of points is

$$K(z,z) = \frac{1}{\pi}.$$

Example 2: Zeros of the i.i.d. power series

Let $\mathcal{E} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ and

$$K(z, w) = \frac{1}{(1 - z\overline{w})^2}. \quad \text{(Bergman kernel)}$$

It was a discovery of B. Virág and Y. Peres that the zeros of the random power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

where a_n are i.i.d. standard complex Gaussian, form a determinantal point process with Bergman kernel.

Some properties

Let $\mathcal{X}$ be a determinantal process with kernel K. Let $A \subseteq \mathcal{E}$.

- $\mathbb{E}[\mathcal{X}(A)] = \int_A K(x,x)\, dx$.
- $\text{Var}(\mathcal{X}(A)) = \int_A \int_{A^c} |K(x,y)|^2\, dx\, dy$.
- $\mathcal{X}(A) \overset{d}{=} \xi_1 + \xi_2 + \ldots$ where ξ_k are independent $\text{Ber}(\lambda_k)$ random variables whose parameters λ_k are the eigenvalues of the integral operator $T_A : L^2(A) \mapsto L^2(A)$ defined by
$$T_A f(x) = \int_A K(x,y) f(y)\, dy.$$
- $\mathbb{P}\{\mathcal{X}(A) = 0\} = \prod_{k \geq 1} (1 - \lambda_k)$.
- Negative association: We state one consequence only.

$$\mathbb{P}\{\mathcal{X}(A) = 0 \text{ and } \mathcal{X}(B) = 0\} \leq \mathbb{P}\{\mathcal{X}(A) = 0\}\mathbb{P}\{X(B) = 0\}.$$

Continuum percolation on the infinite Ginibre ensemble

Let $\mathcal{X}$ be the infinite Ginibre ensemble. Place balls of radius r centered at each point of $\mathcal{X}$. Is there an infinite connected cluster?

As for Poisson process, for small r the answer is No. For large r the answer is yes and in fact there is a unique infinite cluster. [Ghosh, K., Peres 2014].

Key points.

- Negative association helps.
- These point processes are not as flexible as Possion. Important to understand the exact tolerance levels.

A very surprising property of the infinite Ginibre ensemble

Theorem: [S. Ghosh, Y. Peres] Let $\mathcal{X}$ be the infinite Ginibre ensemble. Condition on the configuration of points outside a bounded region A. Then,

1. the number of points in A, i.e., $\mathcal{X}(A)$ is completely determined!
2. Nothing more than that is.

Connectivity question

Suppose we consider the infinite Ginibre ensemble restricted to $[0, n]^2$. Call it $\mathcal{X}_n$. Place balls of radius r_n around each point.

Question: How large should r_n be to ensure that all points are connected.

Answer expected $(\log n)^{\frac{1}{4}}$ [M. K., R. Sundaresan, simple calculation ???]

Contrast with Poisson process where the answer is $(\log n)^{\frac{1}{2}}$.

Key point: In the infinite Ginibre ensemble

$$\mathbb{P}\{\mathcal{X}(D(0; r)) = 0\} \approx e^{-cr^4}.$$

Many questions can be asked, and some perhaps answered...

Many things that we could not touch upon

- Karlin-McGregor determinant formula.
- Generating function formulas for linear statistics.
- Central limit theorems for linear statistics with slow or no variance growth.
- Concentration results of Pemantle and Peres.
- Stationary DPPs with kernel $\hat{\mathbf{1}}_A(x-y)$.
- Tail triviality, ergodicity, mixing and such questions.
- Application to sampling columns of a matrix: Beyond length^2 sampling to sampling pairs of columns according to determinantal probabilities.
- Dynamics: Dyson's Brownian motion. Non-intersecting walks.
- Negative association (Russell Lyons).

And perhaps many more.

A few references

We just refer to three surveys. See references therein for more.

- A. Soshnikov, *Determinantal random point fields*. Russian Math. Surveys 55 (2000), no. 5, 923–975
- R. Lyons, *Determinantal probability measures.* Publ. Math. Inst. Hautes Études Sci. No. 98 (2003), 167–212.
- Hough, J. B., Krishnapur, M., Peres, Y., Virág, B., *Determinantal processes and independence.* Probab. Surv. 3 (2006), 206–229.