Minimal Public Communication for Maximum Rate Secret Key Generation

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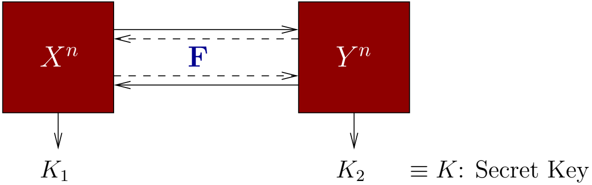
Secret Key Generation



Secret key (SK) K with interactive communication $\mathbf{F}$ satisfies:

$$\Pr(K_1 = K_2 = K) \approx 1 : \quad \text{Recoverability}$$

$$\frac{1}{n} I\left(K \wedge \mathbf{F}\right) \approx 0 : \quad \text{Secrecy}$$

Rate of the secret key $= \frac{1}{n} H(K)$.

Secret key capacity $C =$ maximum achievable rate of a secret key.

[Maurer '93, Ahlswede-Csiszár '93]

$$C = I(X \wedge Y).$$

Communication for SK Capacity

What is the min. rate of $\mathbf{F}$ required for achieving SK capacity?

- Maurer-Ahlswede-Csiszár
  - *Common randomness* (CR) generated: X^n or Y^n.
  - Rate of communication required $= \min\{H(X|Y); H(Y|X)\}$.
  - Decomposition:
    $H(X) = H(X \mid Y) + I(X \wedge Y)$,
    $H(Y) = H(Y \mid X) + I(X \wedge Y)$.
- Csiszár-Narayan
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Q1: Forms of CR for agreeing on optimum rate SK ?

Q2: Does minimum communication rate correspond to "minimum" CR?

Interactive Form of Wyner's Common Information

- Wyner's Common Infomation

  $CI(X \wedge Y) \equiv$ min. rate of a function $L = L\left(X^n, Y^n\right)$ such that

  $$\frac{1}{n} I(X^n \wedge Y^n \mid L) \approx 0.$$

  Defined in the context of source generation and source coding.

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- Interactive Common Information

  Terminals agree on CR J using r-rounds $\mathbf{F}$.

  $CI_i^r(X \wedge Y) \equiv$ min. rate of $(J, \mathbf{F})$ such that

  $$\frac{1}{n} I(X^n \wedge Y^n \mid J, \mathbf{F}) \approx 0.$$

  $CI_i(X \wedge Y) = \lim_{r \to \infty} CI_i^r(X \wedge Y).$

Note: $CI(X \wedge Y) \leq CI_i(X \wedge Y) \leq \min\{H(X); H(Y)\}.$

Minimum Communication for Optimum SK

For minimum rate of communication R_{SK} for optimum rate SK:

We characterize the CR associated with optimum rate SK.

- A CR J recoverable from communication $\mathbf{F} \Rightarrow$ SK of rate $\frac{1}{n}H(J|\mathbf{F})$.
- An optimum rate SK corresponds to a CR J recoverable from $\mathbf{F}$ s.t.

$$\frac{1}{n}I(X^n \wedge Y^n \mid J, \mathbf{F}) \approx 0.$$

  - *For instance:* $J = X^n$, Y^n or (X^n, Y^n).
  - R_{CI} be the min. rate of communication for such CR.

- Shall show: CR of the above form always yields optimum rate SK.

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Theorem

$$R_{SK} = R_{CI} = CI_i(X \wedge Y) - I(X \wedge Y).$$

Minimum Communication for Optimum SK

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- **Lemma:** For each $r \geq 1$:
  - $R_{SK}^r \geq R_{CI}^r \geq R_{SK}^{r+1}$.
  - Decomposition:
    $R_{CI}^r \geq CI_i^r(X \wedge Y) - I(X \wedge Y) \geq R_{CI}^{r+1}$.
- Theorem follows by taking the limit $r \to \infty$ in the Lemma.

Idea of the Proof

- Proof is based on the observation:

$$I(X \wedge Y) \approx \frac{1}{n}\left[I\left(X^n \wedge Y^n \mid J, \mathbf{F}\right) + H(J, \mathbf{F}) - H(\mathbf{F} \mid X^n) - H(\mathbf{F} \mid Y^n)\right].$$

Characterization of the form of CR in optimum SK generation:

$$\frac{1}{n} I\left(X^n \wedge Y^n \mid J, \mathbf{F}\right) \approx 0$$

if and only if

$$I(X \wedge Y) \approx \frac{1}{n}\left[H(J, \mathbf{F}) - \left[H(\mathbf{F} \mid X^n) + H(\mathbf{F} \mid Y^n)\right]\right].$$

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SK rate associated with CR J recoverable from $\mathbf{F}$

Minimum Communication for Optimum SK

Theorem

R_{SK} = min. rate of $\mathbf{F}$ required for optimal rate SK generation.

R_{CI} = min. rate $\mathbf{F}$ required for generating CR J s.t. $X^n \underset{\sim}{\perp\!\!\!\perp} Y^n | (J, \mathbf{F})$.

Then,

$$R_{SK} = R_{CI} = CI_i(X \wedge Y) - I(X \wedge Y).$$

$$CI_i(X \wedge Y) = \lim_{r \to \infty} CI_i^r(X \wedge Y).$$

Characterization of CI_i

- Given rvs X, Y and $r \geq 1$, we have

$$CI_i^r(X \wedge Y) = \min_{U_1,V_1,\ldots,U_r,V_r} I(X,Y \wedge U_1,V_1,...,U_r,V_r),$$

$$U_1 -\circ- X -\circ- Y, \qquad V_1 -\circ- Y, U_1 -\circ- X,$$

$$U_2 -\circ- X, V_1 -\circ- Y, \qquad V_2 -\circ- Y, U_1, V_1 -\circ- X$$

$$\vdots \qquad \vdots$$

$$U_r -\circ- X, U^{r-1}, V^{r-1} -\circ- Y, \qquad V_r -\circ- Y, U^r, V^{r-1} -\circ- X.$$

$$X -\circ- U^r, V^r -\circ- Y$$

- Single-letter expression for CI_i is not (yet) available.

Noninteractive Communication for SK Capacity

Communication from X^n to Y^n:

$$R_{SK}^{one} = \min_{\substack{U -\circ- X -\circ- Y \\ X -\circ- U -\circ- Y}} I(X, Y \wedge U) - I(X \wedge Y)$$

U satisfies: $U -\circ- X -\circ- Y, \quad X -\circ- U -\circ- Y.$

- [Problem 16.26, Csiszár-Körner]

$$\min_{\substack{U -\circ- X -\circ- Y \\ X -\circ- U -\circ- Y}} I(X, Y \wedge U) = \min_{\substack{g(X) s.t. \\ X -\circ- g(X) -\circ- Y}} H(g(X))$$

.

Common Information Quantities

For a pair of rvs X, Y

$$CI_{GC} \leq I(X \wedge Y) \leq CI \leq CI_i \leq CI_i^r \leq CI_i^{r-1} \leq \min\{H(X); H(Y)\}$$

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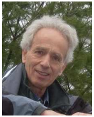

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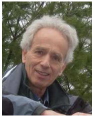

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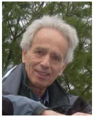

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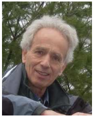

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Interactive Common Information

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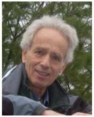

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Interactive Common Information

- CI_i is indeed a new quantity

  For binary rvs X and Y: $CI_i(X \wedge Y) = \min\{H(X); H(Y)\}$.

  For binary symmetric X and Y: $CI(X \wedge Y) < \min\{H(X); H(Y)\}$.