Universal Multiparty Data Exchange

Himanshu Tyagi

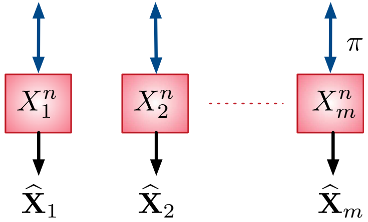
Indian Institute of Science, Bangalore

Shun Watanabe

Tokyo University of Agriculture and Technology

Multiparty Data Exchange

Source model for data exchange



Set of parties, $\mathcal{M} = \{1, ..., m\}$

Observations $X_{\mathcal{M}}^n = \{X_{\mathcal{M}t}\}_{t=1}^n$ are iid with common pmf $\mathrm{P}_{X_{\mathcal{M}}}$

π constitutes an omniscience protocol if $\mathbb{P}\left(\widehat{\mathbf{X}}_1 = ... = \widehat{\mathbf{X}}_m = X_{\mathcal{M}}^n\right) \approx 1$

$R_{\mathsf{CO}}\left(\mathrm{P}_{X_{\mathcal{M}}}\right) \equiv$ Minimum rate of communication for omniscience

Characterization of min. comm. for omniscience

[Csiszár-Narayan 04]

$$R_{\mathsf{CO}}\left(\mathrm{P}_{X_{\mathcal{M}}}\right) = \min_{(R_1,\ldots,R_m)\in\mathcal{R}_{\mathsf{CO}}} \sum_{i=1}^{m} R_i,$$

where

$$\mathcal{R}_{\mathsf{CO}} = \{(R_1, \ldots, R_m) : \sum_{i\in B} R_i \geq H(X_B|X_{B^c}), \quad \forall\, B \subsetneq \mathcal{M}\}$$

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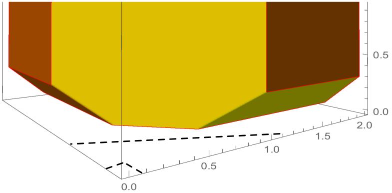

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[Chan-Zheng 10]

$$\min_{(R_1,\ldots,R_m)\in\mathcal{R}_{\mathsf{CO}}} \sum_{i=1}^{m} R_i = \max_{\sigma\in\Sigma(\mathcal{M})} \frac{1}{|\sigma|-1}\mathbb{H}_\sigma,$$

where

$$\mathbb{H}_\sigma = \sum_{i=1}^{|\sigma|} H(X_{\mathcal{M}}|X_{\sigma_i})$$

Our work: A universal protocol

Given $(\mathbf{x}_1, ..., \mathbf{x}_m)$, enable data exchange using

roughly $nR_{\mathtt{CO}}\left(\mathrm{P}_{\mathbf{x}_{\mathcal{M}}}\right)$ bits of communication

Building a Universal Protocol

A basic building block

How should we send $\mathbf{x}$ to $\mathbf{y}$?

[Shulman-Feder 02, Yang-He 10, Braverman-Rao 11]

- Incrementally send $n\Delta$ bits of random hash of $\mathbf{x}$
- Use a variant of "minimum conditional entropy" decoder:

  Find the type $\mathrm{P}_{\overline{X}\,\overline{Y}}$ s.t.

  1. $\mathrm{P}_{\overline{Y}} = \mathrm{P}_{\mathbf{y}}$ and $R \geq H(\overline{X}\,\overline{Y}) - H(\overline{Y})$
  2. $\exists$ unique $\mathbf{x}$ of conditional type $\mathrm{P}_{\overline{X}|\overline{Y}}$ given $\mathbf{y}$

     and consistent with hash values

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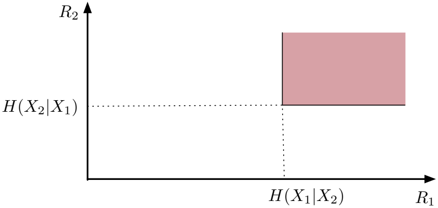
Shall use this to send $\mathbf{x}_{A\setminus\{i\}}$ to $\mathbf{x}_i$

Ideal assumptions: Oracle model

- *Continuous rate:* Rate can be increased continuously
- *Ideal decoder:* An ideal decoder with following features is available
  1. Returns correct $\mathbf{x}_A$, $A \subset \mathcal{M}$, as soon as $(R_i, i \in A) \in \mathcal{R}_{\texttt{CO}}(A)$
  2. If the condition above does not hold for any A, returns a NACK

Protocol for two parties

$$\mathcal{R}_{\mathsf{CO}}(\mathrm{P}_{X_1 X_2}) = \{(R_1, R_2) : R_i \geq H(X_1, X_2 | X_i), i \in \{1, 2\}\}$$



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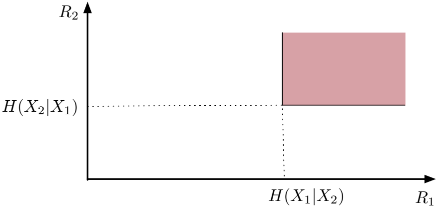

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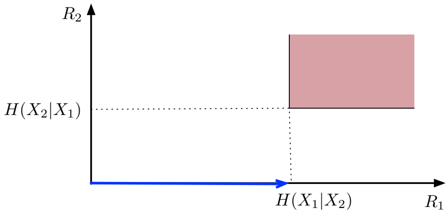

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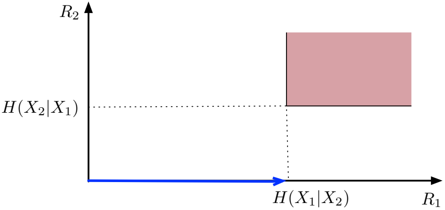

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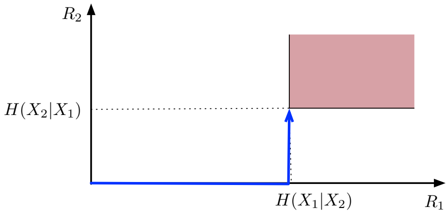

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- Who should start communicating?

- When to start communicating?

- How to increase the rates?

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  Maintain $H(\mathbf{x}_i) - H(\mathbf{x}_j)$ for all communicating parties i, j

- How to increase the rates?

Example 1 $(m = 2)$

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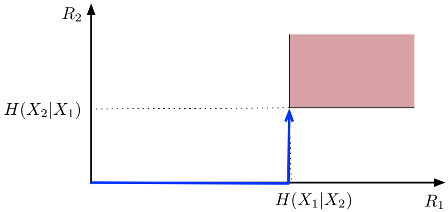

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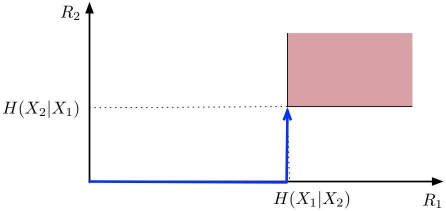

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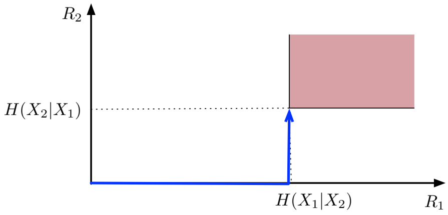

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Universal Protocol 2:

1. Parties compute their types (empirical distributions) $\mathrm{P}_{\mathbf{x}_i}$ and share

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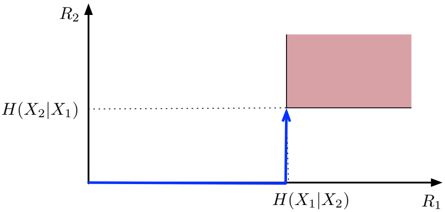

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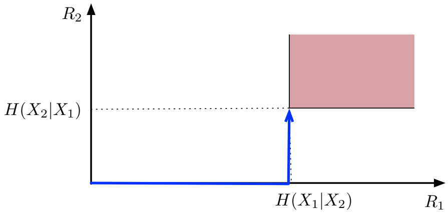

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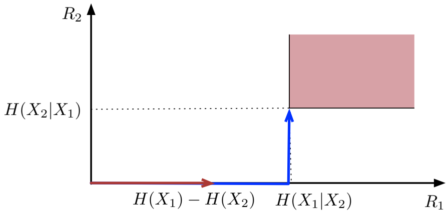

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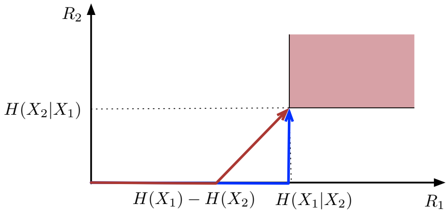

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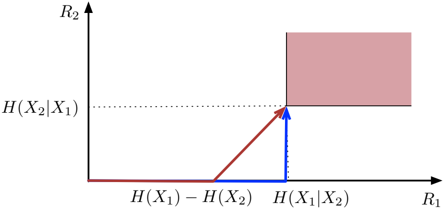

Observation 1: $R_1^* - R_2^* = H(X_1|X_2) - H(X_2|X_1) = H(X_1) - H(X_2)$

Observation 2: Both parties will simultaneously decode each other

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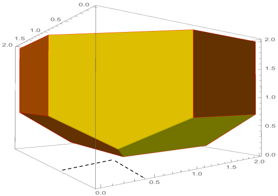
Example 2 ($m = 3$)

$X_1 \sim \texttt{Ber}(1/2)$, $X_3 \sim \texttt{Ber}(q)$, $X_2 = X_1 \oplus X_3$, $h(q) > 1/2$

- Finest partition is dominant
- The unique optimal rate assignment is given by $\mathbf{R}^* = (1/2, 1/2, h(q) - 1/2)$



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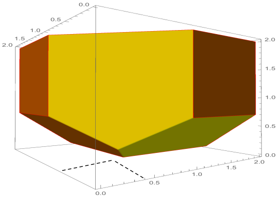

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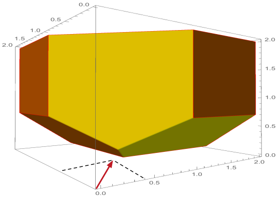

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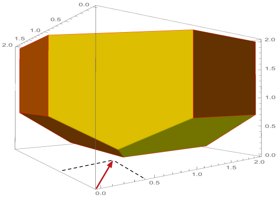

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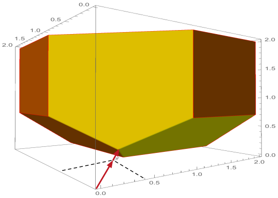

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If parties in A attain "local omniscience," they start behaving as one and increment the rates at slope $1/|A|$

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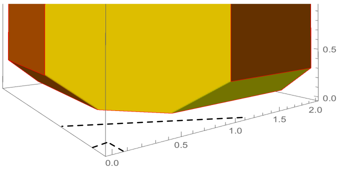

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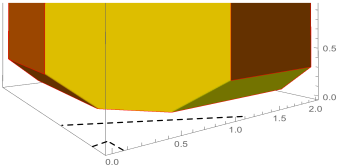

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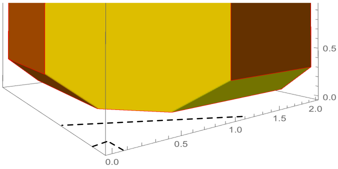

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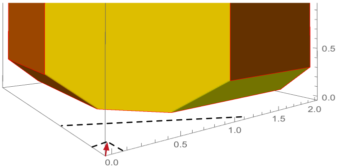

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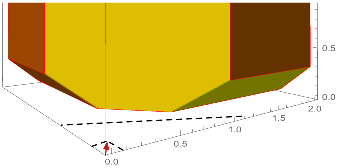

1. Parties 1 and 2 start at slope 1
2. They attain local omniscience when $R_1 = h(q) = R_2$
3. Parties 1 and 2 increase rates at slope $1/2$

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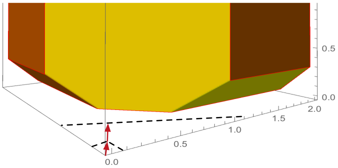

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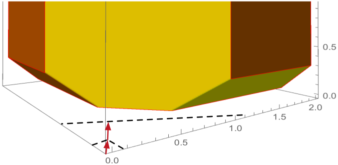
$W_1, W_2 \sim \texttt{Ber}(1/2), \quad V_1, V_2 \sim \texttt{Ber}(q), \quad q < 1/2$

$X_1 = (W_1, W_2), \quad X_2 = (W_1 \oplus V_1, W_2), \quad X_3 = W_2 \oplus V_2$

- Partition $\{12|3\}$ is dominant



1. Parties 1 and 2 start at slope 1
2. They attain local omniscience when $R_1 = h(q) = R_2$
3. Parties 1 and 2 increase rates at slope $1/2$
4. Parties 3 starts when $R_1 + R_2 - R_3 = H(X_1, X_2) - H(X_3) = 1 + h(q)$

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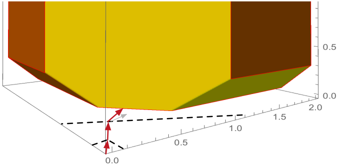

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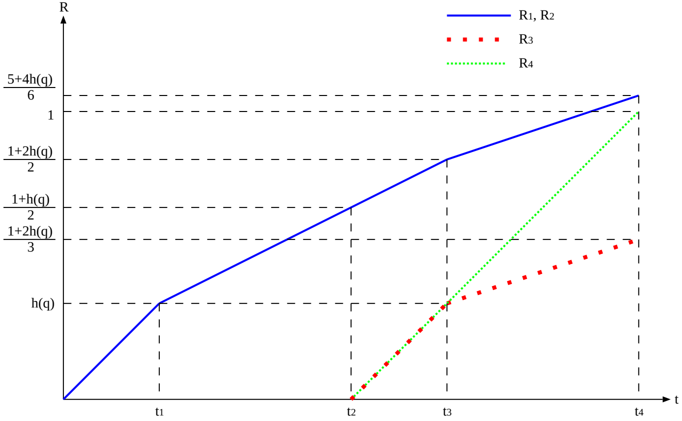
Example 4 ($m = 4$)

$W_1, W_2, W_3 \sim \texttt{Ber}(1/2), \quad V_1, V_2 \sim \texttt{Ber}(q), \qquad q < 1/2$

$X_1 = (W_1, W_2), \quad X_2 = (W_1 \oplus V_1, W_2), \quad X_3 = W_2 \oplus V_2, \quad X_4 = W_3$

- Partition $\{123|4\}$ is dominant



A Universal Protocol for Multiparty Data Exchange

The OMN subroutine

$\texttt{OMN}(\sigma, \mathbf{H}, \mathbf{R})$

Inputs

$\mathbf{H} = (H_{\sigma_1}, ..., H_{\sigma_k})$ is a decreasing sequence

$\mathbf{R} = (R_1, ..., R_m)$

Outputs

$\mathcal{O}$: the set of subsets that attain omniscience

$\mathbf{R}^{\text{out}}$: rates of communication when OMN terminates

Execution

While all decoders output NACK

1. All parties with $R_i > 0$, $i \in \sigma_l$, increase their rates at "slope" $1/|\sigma_l|$
2. A new party $j \equiv \sigma_j$ starts communicating if

$$R_{\sigma_1} - R_{\sigma_j} = H_{\sigma_1} - H_{\sigma_j}$$

3. Each party is running the ideal decoder

Main observation: The recursive structure of OMN

If `OMN` is called with a valid rate vector $\mathbf{R}$

If a new subset A attains local omniscience:

(i) A is of the form $\{\sigma_{i_1}, ..., \sigma_{i_l}\}$;

(ii) $\mathbf{R}^{\text{out}}$ is as if the parties in A were together from the start

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(ii) $\mathbf{R}^{\text{out}}$ is as if the parties in A were together from the start

The sum rate R_A is given by

$$R_A = \mathbb{H}_{\sigma_f(A)}(A) = \frac{1}{l-1} \sum_{j=1}^{l} H(X_A | X_{\sigma_{i_j}})$$

Protocol under ideal assumptions

Initialization

$$\mathbf{R} = (0, -1, -1, ..., -1)$$

$$\mathbf{H} = (H(\mathrm{P}_{\mathbf{x}_1}), ..., H(\mathrm{P}_{\mathbf{x}_m}))$$

$$\sigma = \sigma_f(\mathcal{M})$$

Execution

While omniscience is not attained

1. Call $\mathtt{OMN}(\sigma, \mathbf{H}, \mathbf{R})$; let output be $\mathcal{O}$ and $\mathbf{R}^{\text{out}}$
2. **Update:**

   $\mathbf{R} = \mathbf{R}^{\text{out}}$

   $\sigma =$ parts consist of subsets that have attained local omniscience

   $\mathbf{H} = (H_{\sigma_1}, ..., H_{\sigma_k})$
3. Go to step 1

The Fact of the Matter

Individual sequence performance

Theorem

For every $\Delta > 0$ and every sequence $\mathbf{x}_{\mathcal{M}}$, the probability of error for our protocol is bounded above by

$$C_1 \left(\frac{\log |\mathcal{X}_{\mathcal{M}}|}{\Delta} + m \right) p(n) 2^{-n\Delta}.$$

Furthermore, if an error does not occur, the number of bits communicated by the protocol for input $\mathbf{x}_{\mathcal{M}}$ is bounded above by

$$nR_{\mathsf{CO}}(\mathcal{M}|\mathrm{P}_{\mathbf{x}_{\mathcal{M}}}) + nC_2\Delta + C_3 \left(\frac{\log |\mathcal{X}_{\mathcal{M}}|}{\Delta} + m \right) + C_4 \log n.$$