Bandlimited Signal Reconstruction With Samples Obtained at Unknown but Statistically Distributed Locations

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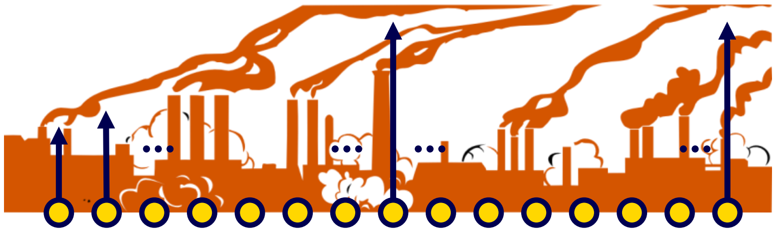
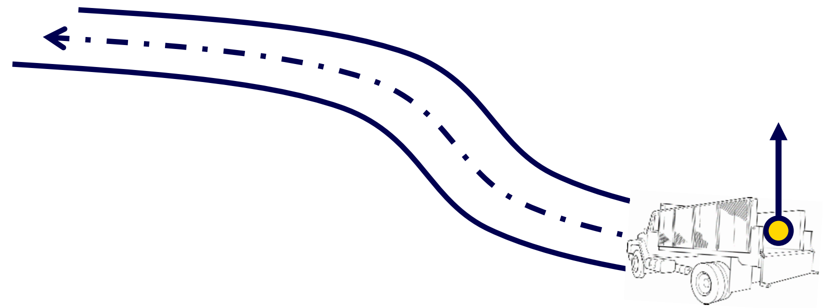
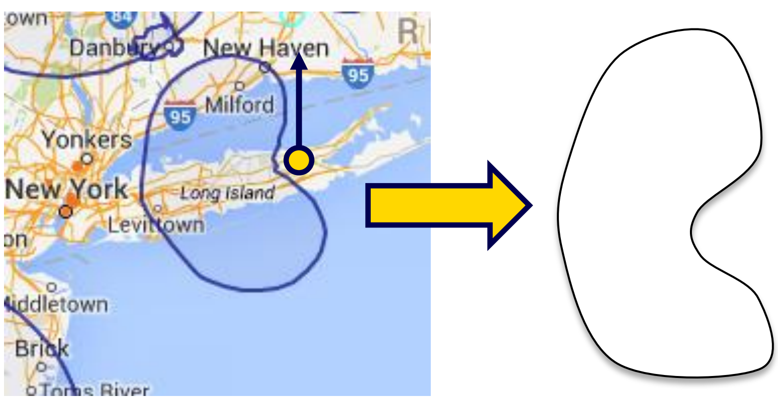
Spatial sampling is getting everywhere now



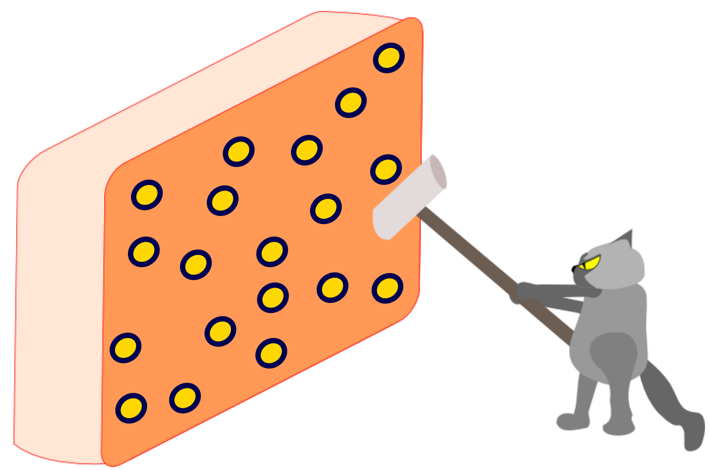
Emission monitoring with sensors



Sampling along a path with vehicle

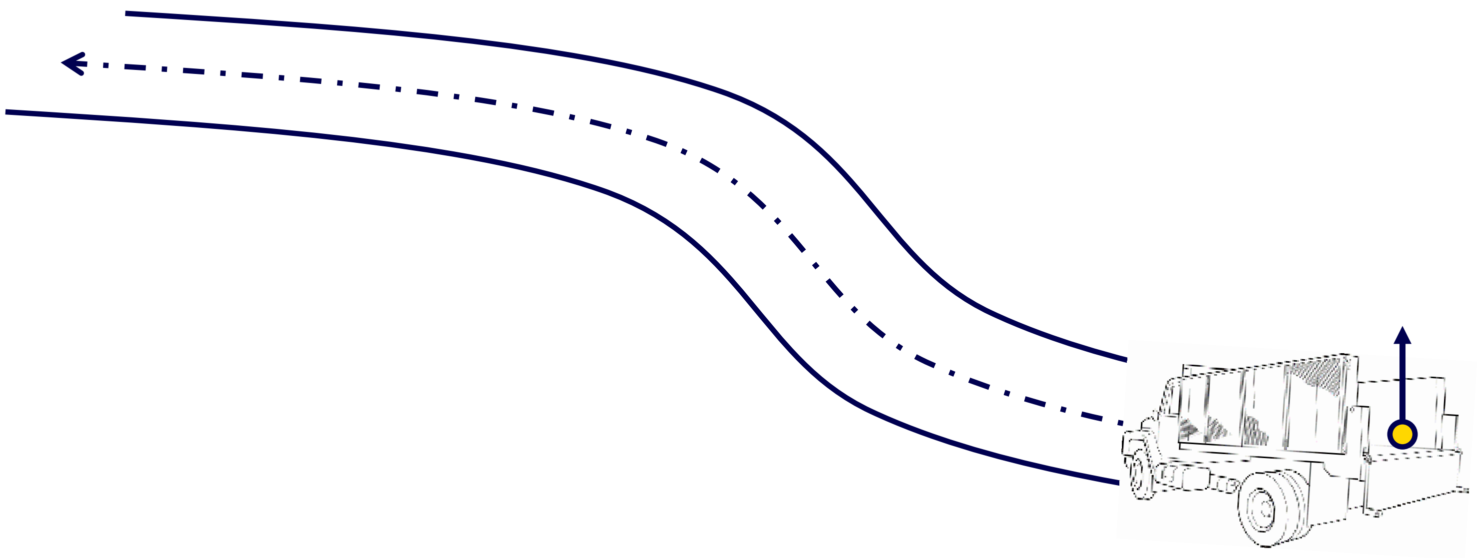


Coverage region for TV transmitters



Randomly sprayed smart-dust/paint

Sampling using vehicles: mobile-sampling

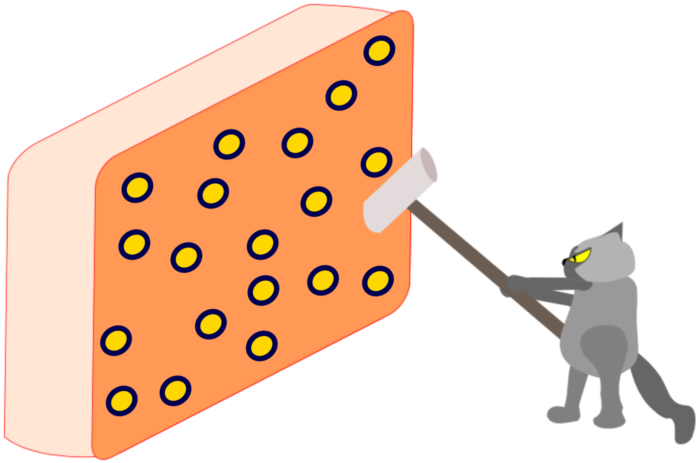


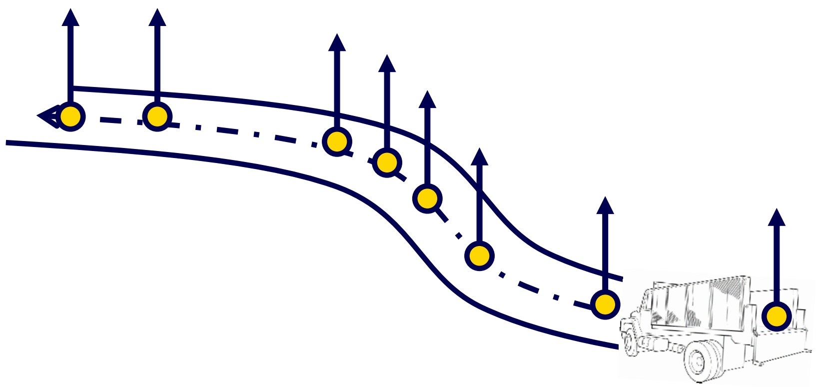
Sampling with moving vehicles can be (has been) used:

◊ **Difficulties:** nonuniform vehicle speeds, *imprecise locations*, noise, quantization, temporal variation

◊ **Aid to estimation:** bandlimitedness, smoothness, **oversampling**

Spatial sampling with “unknown” location

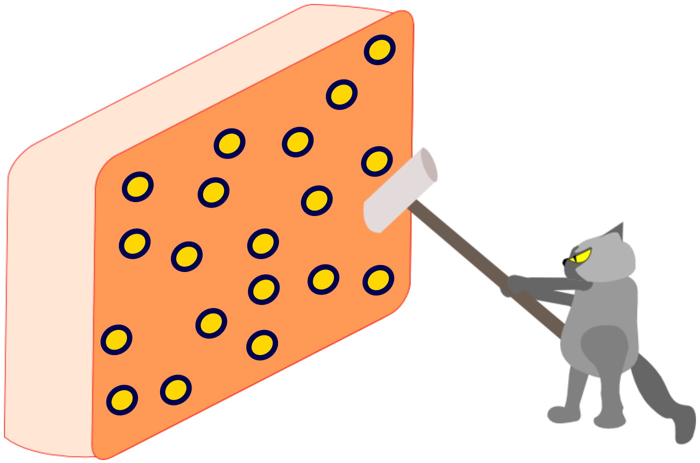


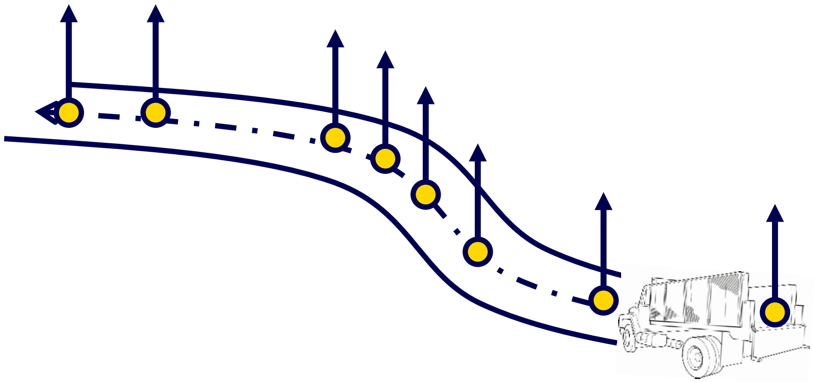


Sampling with fixed array of randomly deployed sensors:

- ◊ **Difficulties:** *nonuniform unknown locations*, quantization, noise, temporal variation
- ◊ **Aid to estimation:** bandlimitedness, smoothness, **oversampling**

Unknown locations: to order or not to order





unknown locations

(more difficult problem)

ordered unknown locations

In these approaches, any effort/cost needed to localize the sensors is saved (which, by the way, is expensive!)

Summary of the talk

Consider **spatially bandlimited fields**, i.e., fields with finite number of non-zero Fourier series coefficients

With **unknown but statistically distributed** sensing locations we address the following problems:

- ◊ Unique determination of the field without order information on samples
- ◊ Field estimation with order of samples known in the presence of measurement noise

Related work

◊ Recovery of (narrowband) discrete-time bandlimited signals from samples taken at unknown locations **[Marziliano and Vetterli'2000]**

◊ Recovery of a bandlimited signal from a finite number of ordered nonuniform samples at unknown sampling locations **[Browning'2007]**.

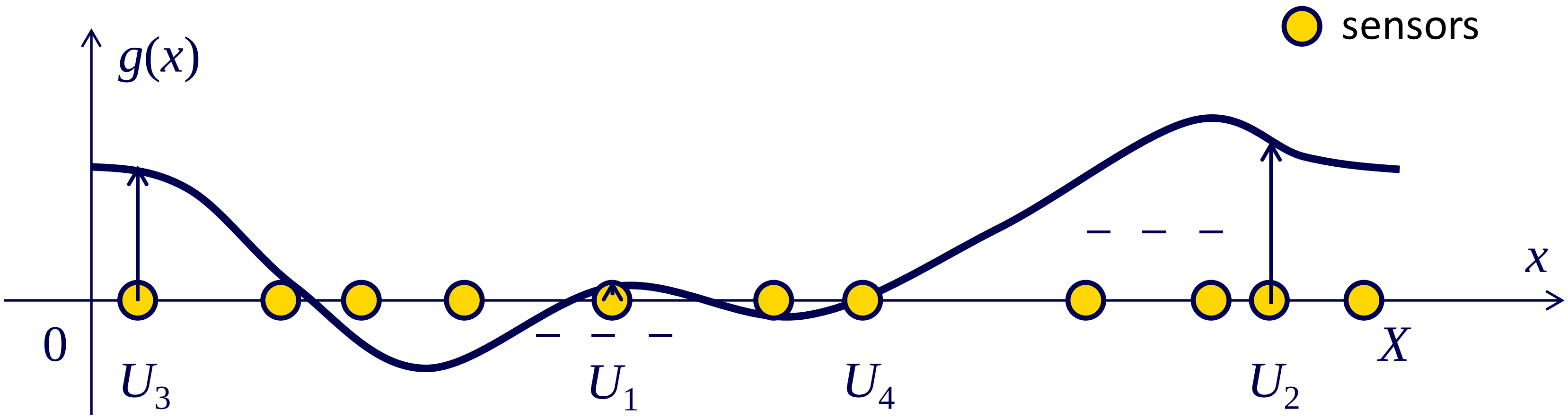
◊ Estimation of periodic bandlimited signals in the presence of random sampling location under two models **[Nordio, Chiasserini, and Viterbo'2008]**

- Reconstruction of bandlimited signal affected by noise at random but known locations
- Estimation of bandlimited signal from noisy samples on a location set obtained by random perturbation of equi-spaced deterministic grid

Organization

- ◊ Sampling model, field model, and distortion
- ◊ Field estimation without any knowledge of sampling location
- ◊ Field estimation with order of samples known in the presence of measurement noise
- ◊ Future work

Field and sensor-locations models

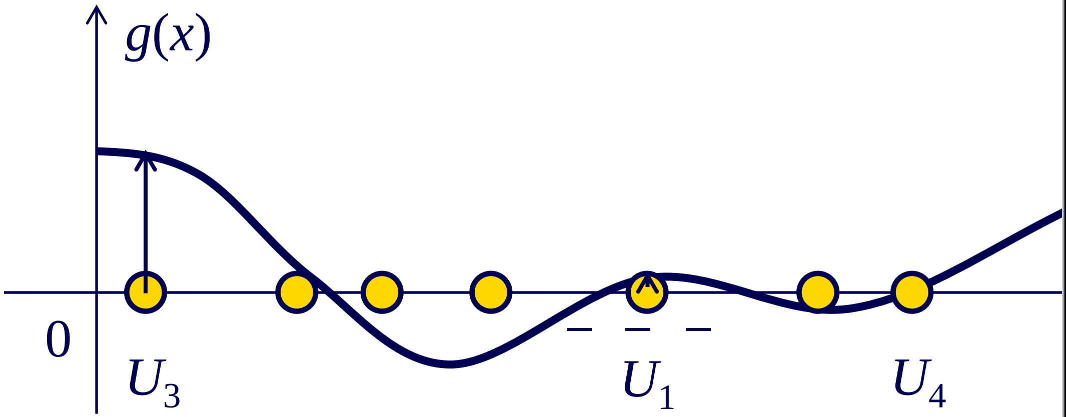


Sensor locations are **unknown** but their statistical distribution is known. For this work, $U_1^n = (U_1, U_2, \ldots, U_n)$ are i.i.d. Unif$[0,X]$

We assume that a periodic extension of the field $g(x)$ is bandlimited, that is, $g(x)$ is given by a finite number of Fourier series coefficients, (WLOG) $|g(x)| \leq 1$, and $X = 1$

$$g(x) = \sum_{n=-b}^{b} a_n \exp(j2\pi nx)$$

Field and sensor-locations models



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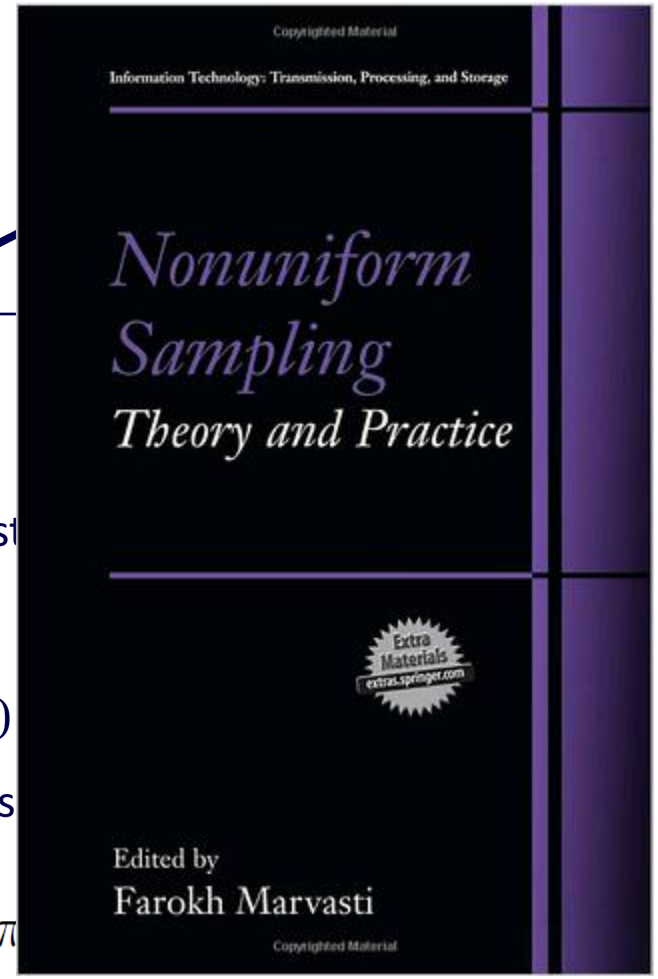
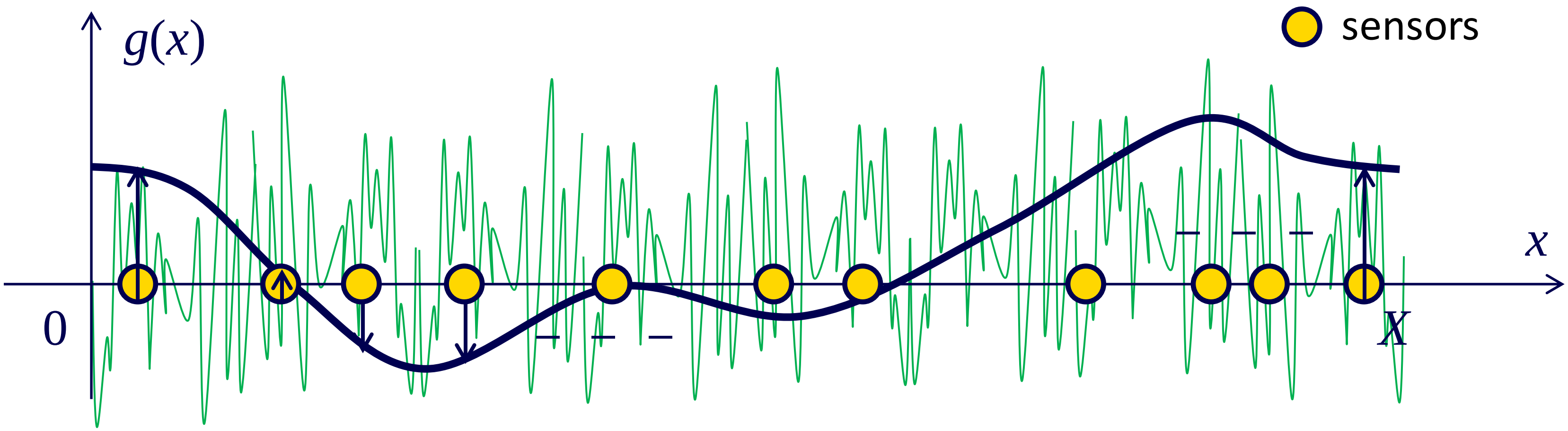


Measurement noise model



For the same model on random deployment of sensors for sampling the field with unknown sensor locations, two cases will be addressed:

- ◊ Sensor measurements are not affected by noise with unordered samples
- ◊ Sensor measurements are affected by additive independent noise with finite variance, with ordered samples

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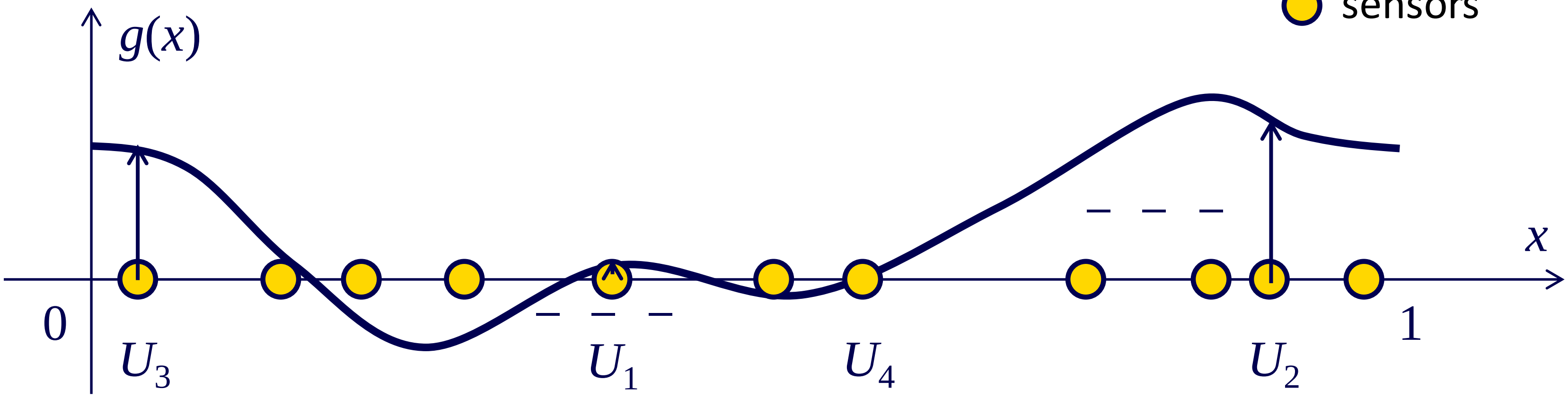
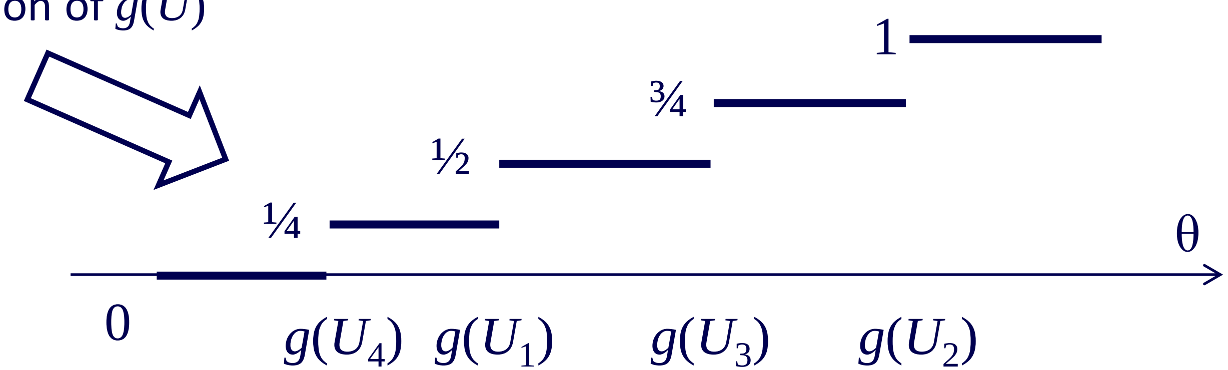
It is impossible to infer $g(x)$ from $g(U_1^\infty)$



Effectively, we are just collecting the empirical distribution or histogram of $g(U_1)$, $g(U_2)$, …, $g(U_n)$ and, in the limit of large n, the task is to estimate $g(x)$ from the distribution of $g(U)$



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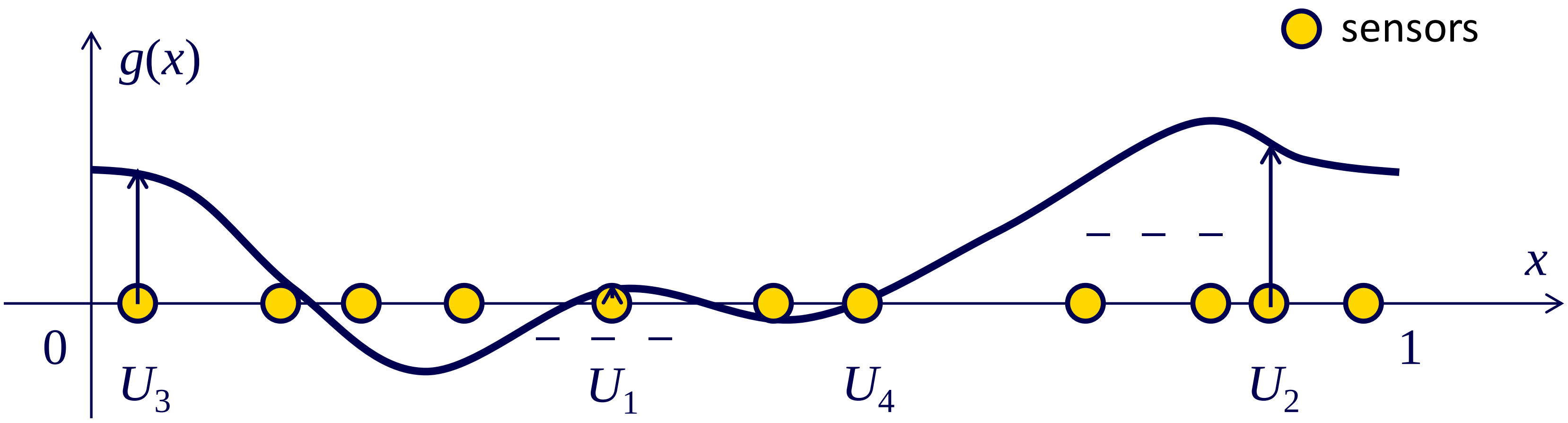


Consider the statistic

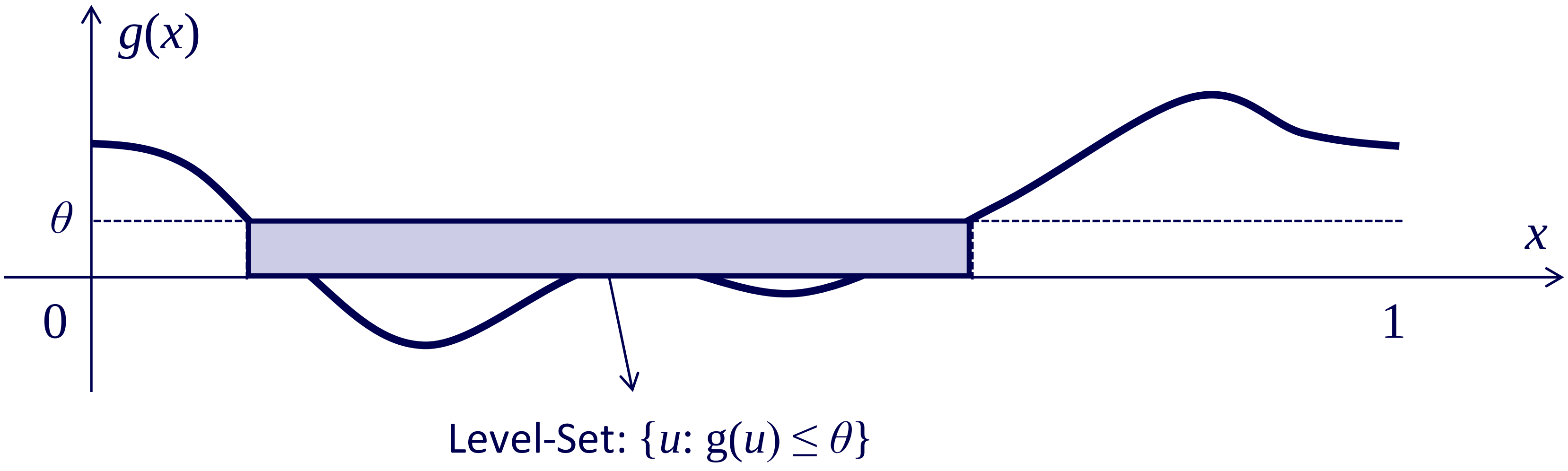
$$F_{g,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[g(U_i) \leq \theta]$$

◊ Then $F_{g,n}(\theta)$, x in set of reals and $g(U_1)$, $g(U_2)$, …, $g(U_n)$ are statistically equivalent

◊ By the Glivenko Cantelli theorem, $F_{g,n}(\theta)$ converges almost surely to Prob$(g(U) \leq \theta)$ for each θ in set of real numbers **[van der Vaart'1998]**

Intuition into the limit of $F_{g,n}$

So what does Prob($g(U) \leq \theta$), for x in set of real numbers, looks like?



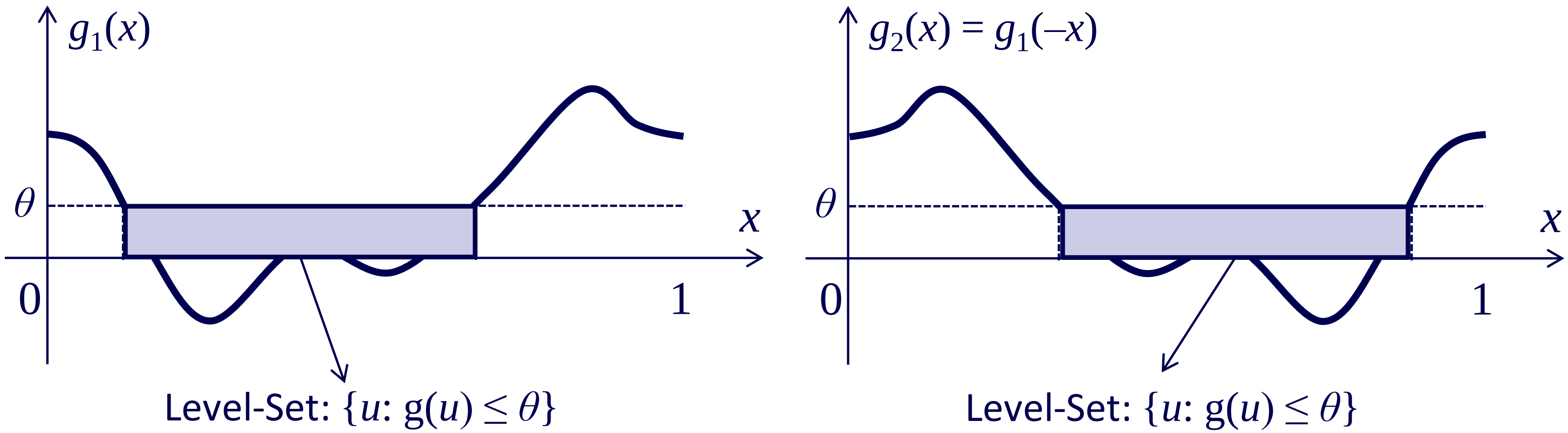
◊ Prob($g(U) \leq \theta$) for each θ is the probability of U belonging in the level-set. Thus, it is simply the length (measure) of level-set

◊ We will now illustrate that two different fields $g_1(x) \neq g_2(x)$ can still lead to

Prob($g_1(U) \leq \theta$) = Prob($g_2(U) \leq \theta$)

Graphical proof of first result

$g_1(x) \neq g_2(x)$ does not imply Prob$(g_1(U) \leq \theta)$ = Prob$(g_2(U) \leq \theta)$



◊ The length (measure) of the level-sets is the same in the two cases for every θ

◊ Glivenko Cantelli theorem's limit, is the same for two different signals. Thus, the observed samples alone do not lead to a unique reconstruction of the field

Can we infer $g(x)$ from $g(T_1^\infty)$?

Consider the setup where the field $g(x)$ is sampled with a non-uniform distribution T. Can we design T to obtain $g(x)$ uniquely?

This is a "**design a distribution**" problem as illustrated next! Key idea is to **break the symmetry** present in the uniform distribution

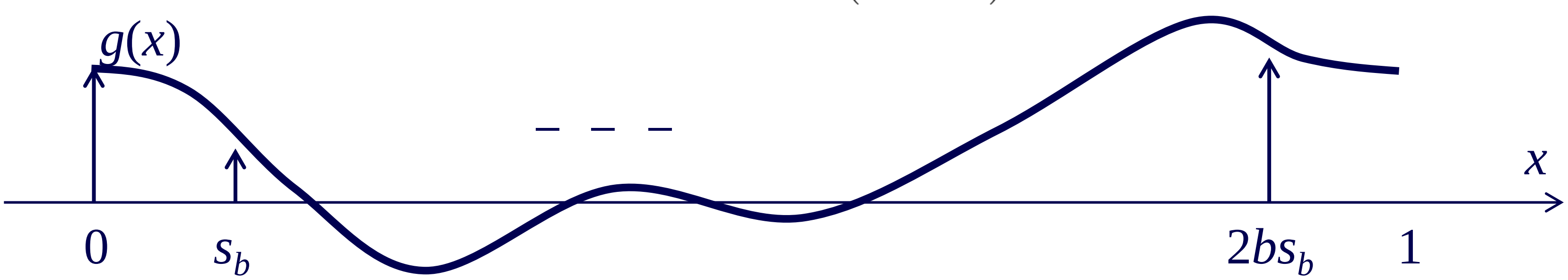
Field samples and the Fourier series

From $(2b+1)$ equi-spaced samples of the field, the $(2b+1)$ Fourier series coefficients (and hence the field) can be obtained as follows

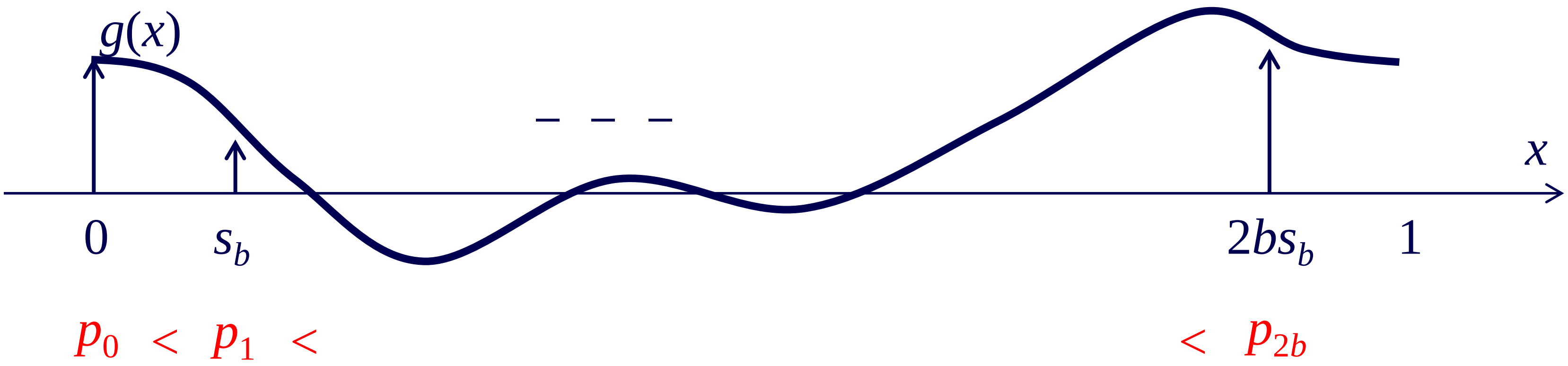
$$\begin{bmatrix} g(0) \\ g(s_b) \\ \vdots \\ g(2bs_b) \end{bmatrix} = \begin{bmatrix} 1 & \dots & 1 \\ \phi_{-b} & \dots & \phi_b \\ \vdots & & \vdots \\ (\phi_{-b})^{2b} & \dots & (\phi_b)^{2b} \end{bmatrix} \begin{bmatrix} a_{-b} \\ a_{-b+1} \\ \vdots \\ a_b \end{bmatrix}$$

where $s_b = 1/(2b+1)$ and $\phi_b = \exp(j2\pi k s_b) = \exp(j2\pi k/(2b+1))$. In matrix notation and upon inversion

$$\vec{a} = (\Phi_b)^{-1}\vec{g} = \frac{1}{(2b+1)}\Phi_b^{\dagger}\vec{g}$$

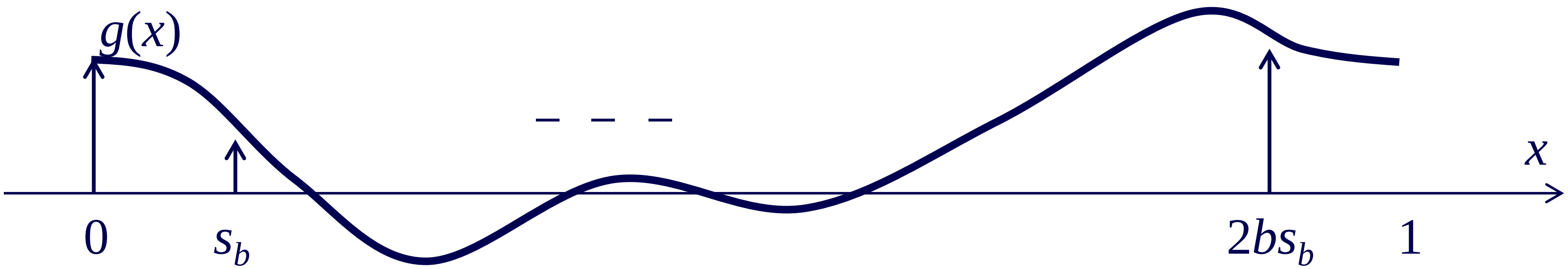


A bandlimited spatial field "detection" problem



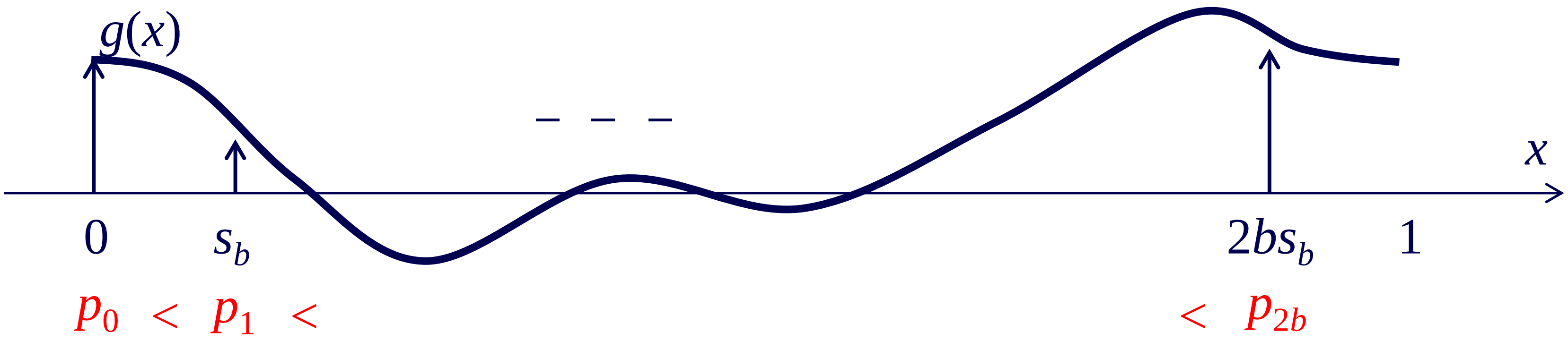
- Assume that T takes values $0, s_b, 2s_b, \ldots, 2bs_b$
- The samples $g(T_1), \ldots, g(T_n)$ are available, but the values of $T_1, \ldots, T_n$ are not known
- Assume that the values $g(0), g(s_b), \ldots, g(2bs_b)$ are distinct
- Design a distribution $(p_0, p_1, \ldots, p_{2b})$ to maximize correct detection of samples (hence the field)

Field detection as an optimization problem



- ◊ N_0, number of times $g(0)$ is observed; N_1, number of times $g(s_b)$ is observed, …, N_{2b}, number of times $g(2bs_b)$ is observed
- ◊ Error doesn't happen when $0 < N_0 < N_1 < \ldots < N_{2b}$
- ◊ To prevent error, it is desirable to space apart $(p_0, p_1, \ldots, p_{2b})$
- ◊ On the other hand $p_0 + p_1 + \ldots + p_{2b} = 1$

Main result of distribution optimization



Theorem: Using Sanov's theorem and large-deviation theory we can show that the optimal distribution for minimizing error probability of field detection is given by **[Mallick-Kumar (submitted to TSP)]**

$$p_i = (i+1)^2 \frac{3}{(b+1)(2b+1)(4b+3)}$$

Essentially, $\mathrm{sqrt}\{p_0\} = \mathrm{sqrt}\{p_1\} - \mathrm{sqrt}\{p_0\} = \mathrm{sqrt}\{p_2\} - \mathrm{sqrt}\{p_1\} = \ldots$

Organization

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- ◊ Field estimation with order of samples known in the presence of measurement noise
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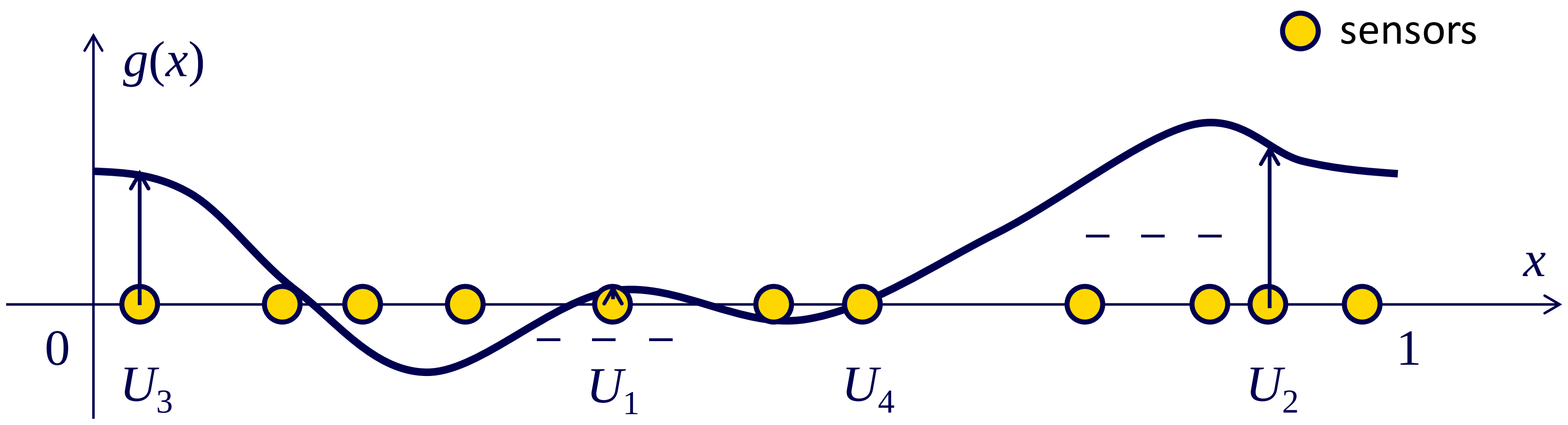
The distortion criterion



$(g(U_1), g(U_2), \ldots, g(U_n))$ is collected without the knowledge of $(U_1, U_2, \ldots, U_n)$

We wish to estimate $g(x)$ and measure the performance of estimate against the average mean-squared error, i.e., if $\hat{G}(x)$ is the estimate then

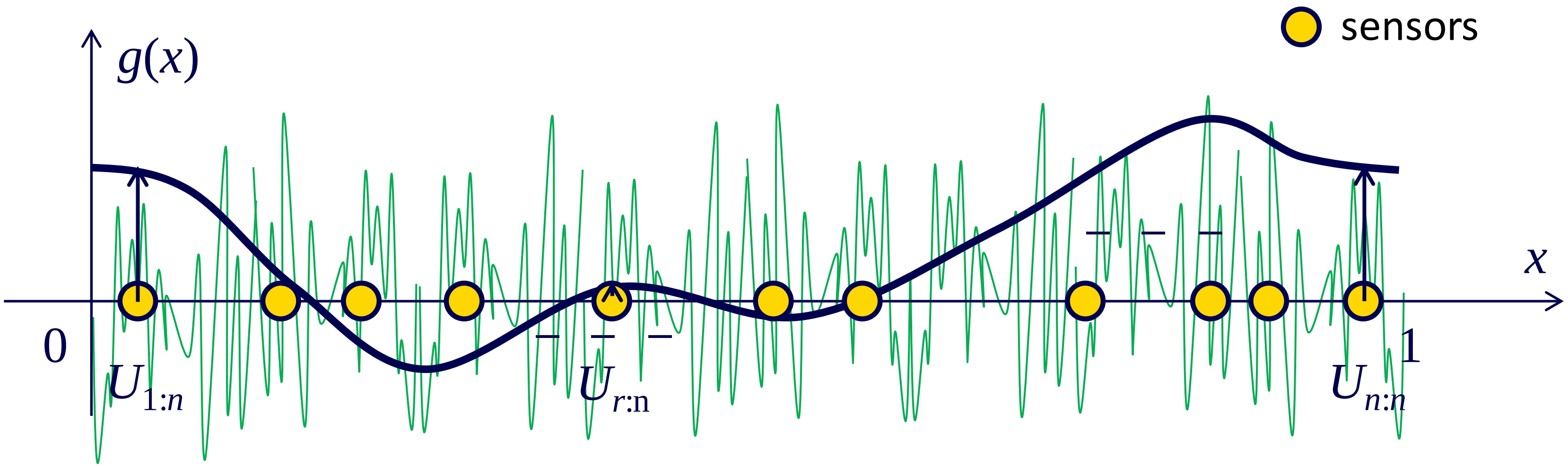
$$D := \|\widehat{G} - g\|_2^2 := \int_0^1 |\widehat{G}(x) - g(x)|^2 \mathrm{d}x$$

Ordered samples in additive indep. noise

◊ If the **order** (left to right) of sample locations is known and field is affected by independent measurement noise, a consistent estimate $\hat{G}(x)$ for the field of interest can be obtained as follows

$$Y(x) = g(x) + W(x) = \sum_{n=-b}^{b} a_n \exp(j2\pi nx) + W(x)$$

◊ Due to bandlimitedness, there are $(2b+1)$ parameters to be learned or estimated



Fourier series coefficient estimates

◊ It is assumed that b is known

◊ The ordered samples $Y(U_{1:n})$, …, $Y(U_{n:n})$ are available, but the values of $U_{1:n}$, …, $U_{n:n}$ are not known. The following estimate can be used

$$\begin{aligned}
\widehat{A}_i &= \frac{1}{n}\sum_{\beta=1}^{n} Y(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&= \frac{1}{n}\sum_{\beta=1}^{n} g(U_{\beta:n})\exp(-j2\pi i\beta/n) + \frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&\approx \int_0^1 g(x)\exp(-j2\pi i x)\mathrm{d}x + \frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&= a_i + \frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n)
\end{aligned}$$

Main result

Theorem: Let Fourier series coefficient estimates for $g(x)$ be obtained as

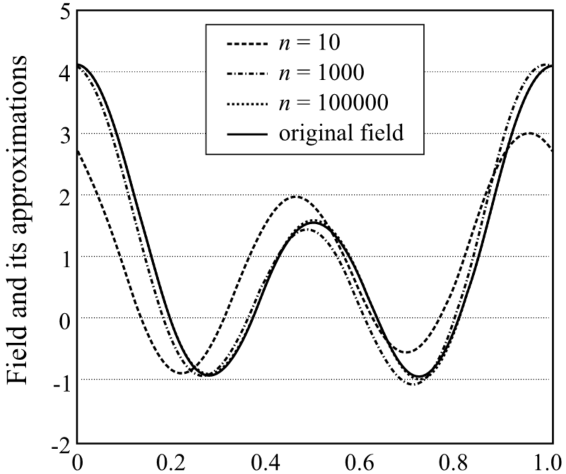
$$\widehat{A}_i = \frac{1}{n}\sum_{\beta=1}^{n} Y(U_{\beta:n})\exp(-j2\pi i\beta/n)$$

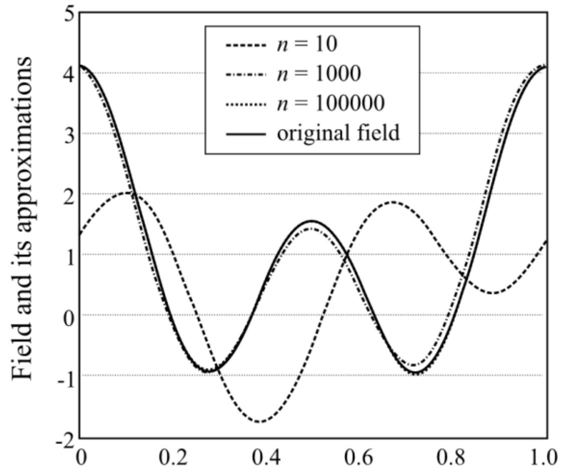
Then the average mean-squared error (distortion) between $g(x)$ and its estimate $G(x)$ with Fourier series coefficients above is bounded by

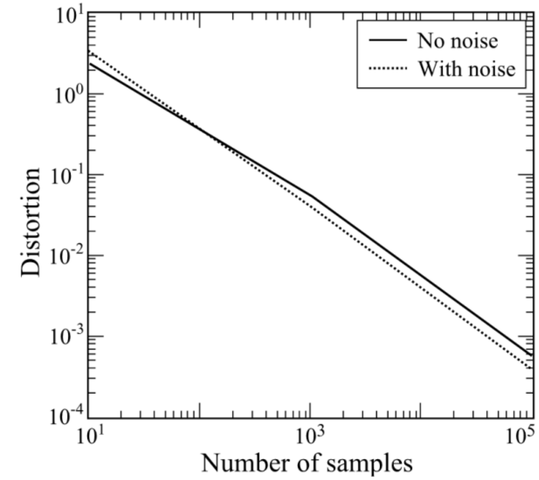
$$\begin{aligned}\mathbb{E}\left|G(x)-g(x)\right|^2 &= \sum_{i=-b}^{b}\mathbb{E}\left|\widehat{A}_i - a_i\right|^2 \\ &\leq (2b+1)\left[\frac{\pi^2 b^2}{n} + O\left(\frac{1}{n\sqrt{n}}\right) + \frac{16\pi^2 b^2}{n^2} + \frac{\sigma^2}{n}\right] \\ &= O\left(\frac{1}{n}\right).\end{aligned}$$

where is the σ^2 variance of the additive noise **[Kumar'2015]**

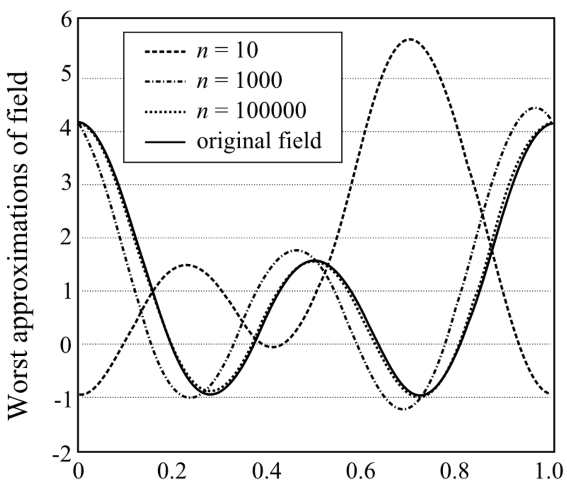
Simulation results

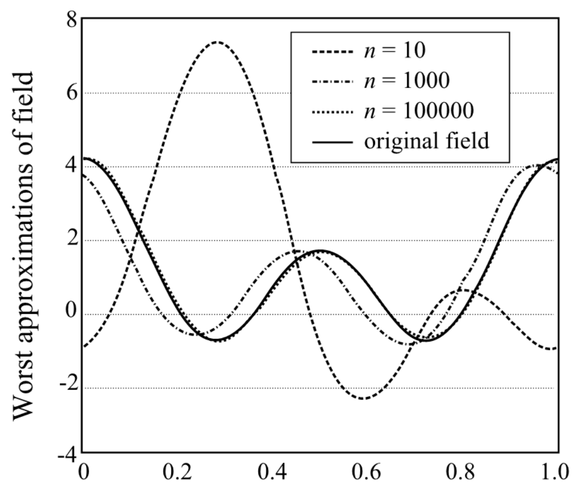


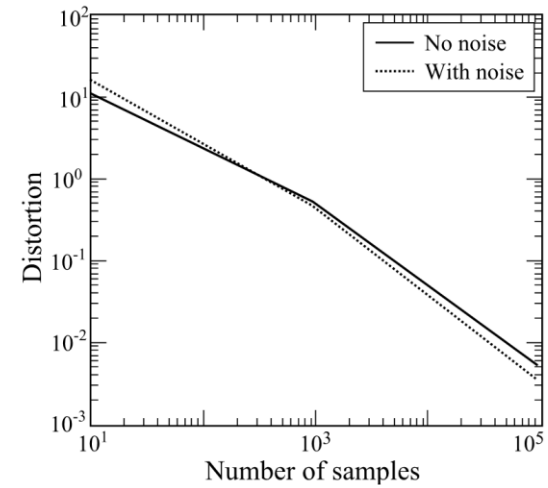




Fourier series coefficients are given by [0.9134, 0.6324, 1.0000, 0.6324, 0.9134]







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Future work

- Extension of these results to more classes of fields (**FRI**, orthogonal spaces, **non-bandlimited** fields)
- **Multidimensional** counterparts
- What is the effect of **quantization**?
- Estimates are not minimum risk. Or, techniques for finding maximum likelihood estimates will be useful
- It is unclear if $O(1/N)$ distortion obtained is optimal
- Lot more …

Selected publications

1. A. Kumar, “On Bandlimited Signal Reconstruction From the Distribution of Unknown Sampling Locations”, IEEE Transactions on Signal Processing
2. A. Mallick and A. Kumar, “”, Submitted to IEEE Transactions on Signal Processing