Asymptotics of the SIR Distribution in General Cellular Networks

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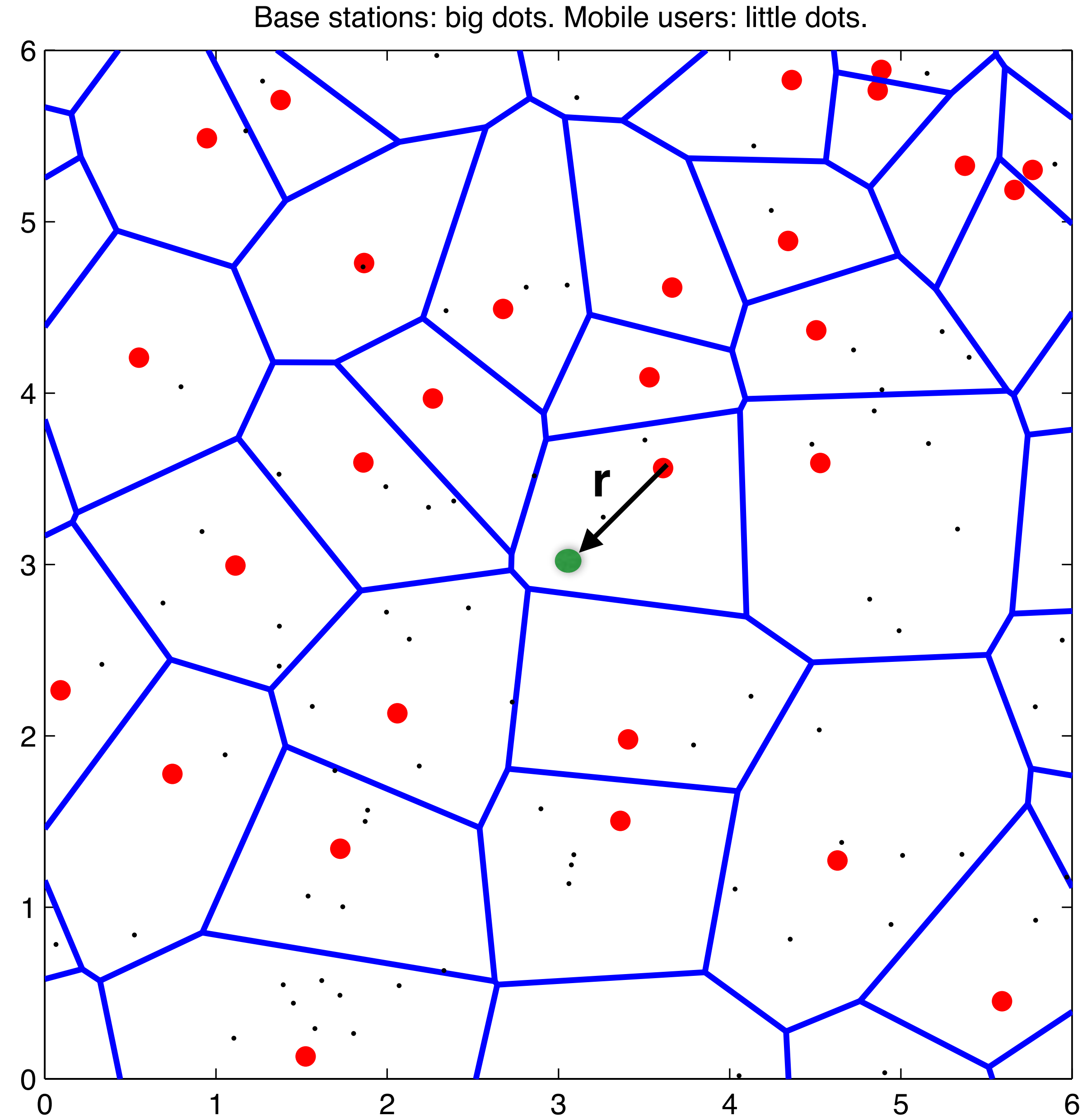
Joint work with Martin Haenggi

Signal-to-interference ratio

$$\mathrm{SIR} = \frac{h_o r^{-\alpha}}{\sum_{y \in \Phi \setminus \{y_o\}} h_y \|y\|^{-\alpha}}$$

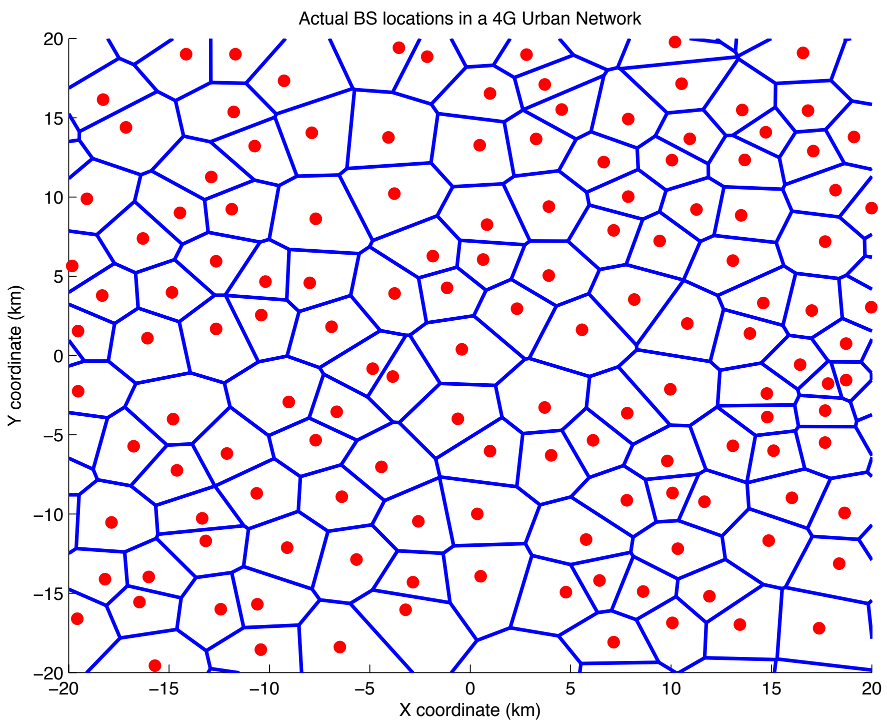


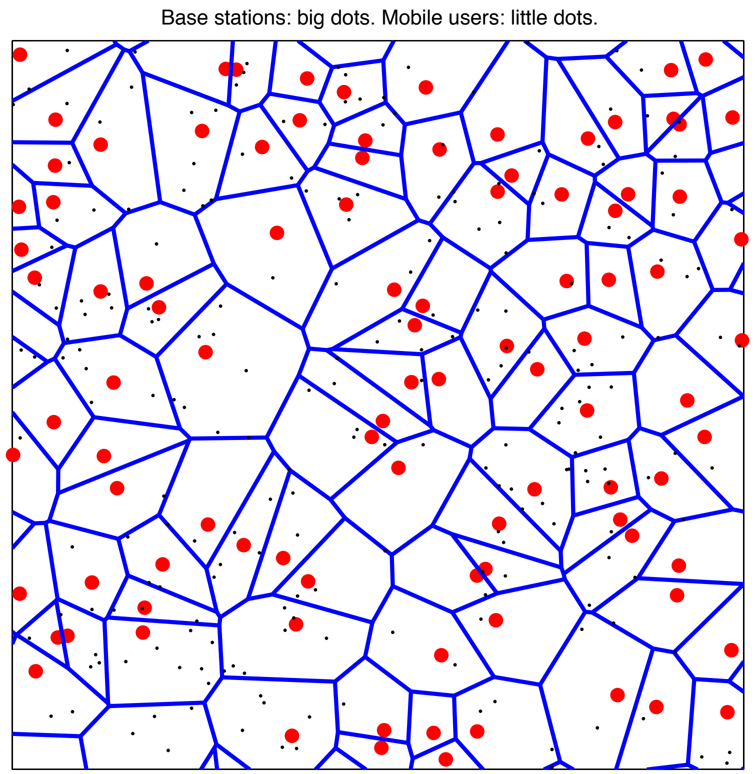
Sources of randomness

- Fading (Rayleigh)
- Base station locations
  - Serving
  - Interferers

For a fixed fading distribution, what can be said about the SIR CDF?

Models for spatial locations





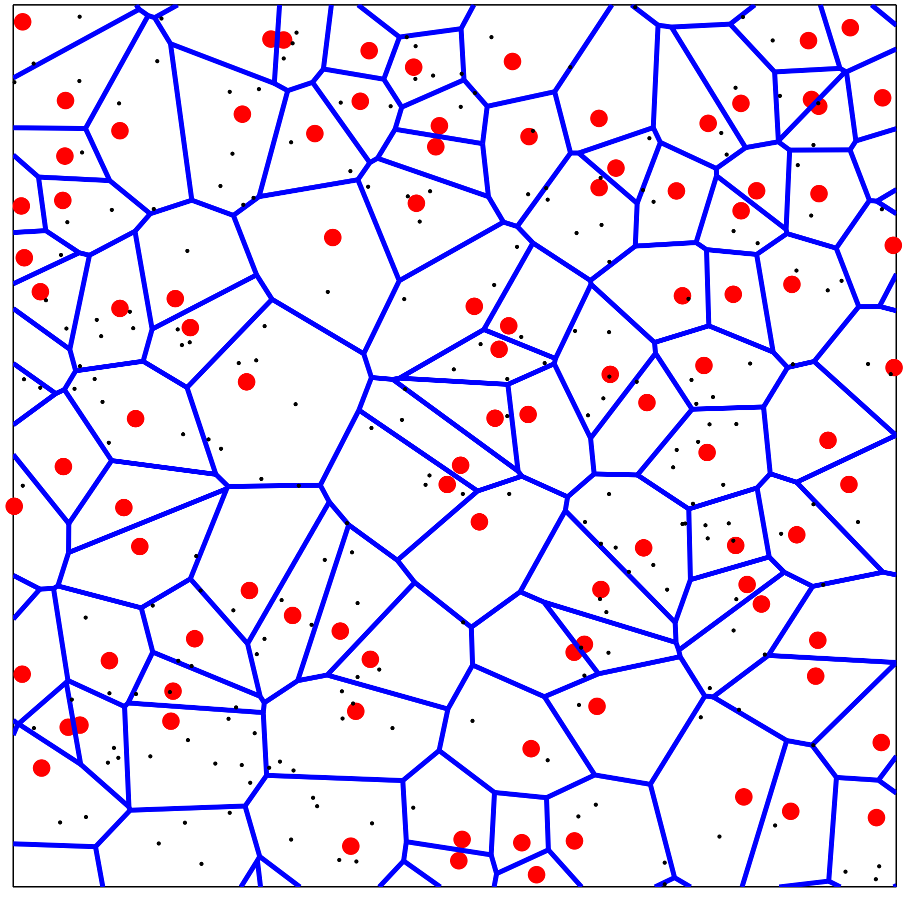
Assumptions

- Stationary point process $\Phi \stackrel{d}{=} \Phi + x$
- Simple and finite point process

Poisson point process

- Easiest to analyze[1] (similar to M/M/1)
- Independence across node locations

$$P_c(T) = \frac{1}{1 + T^{2/\alpha} \int_{T^{-2/\alpha}}^{\infty} \frac{1}{1+u^{\alpha/2}} du}$$



Tractable and most work done using PPP

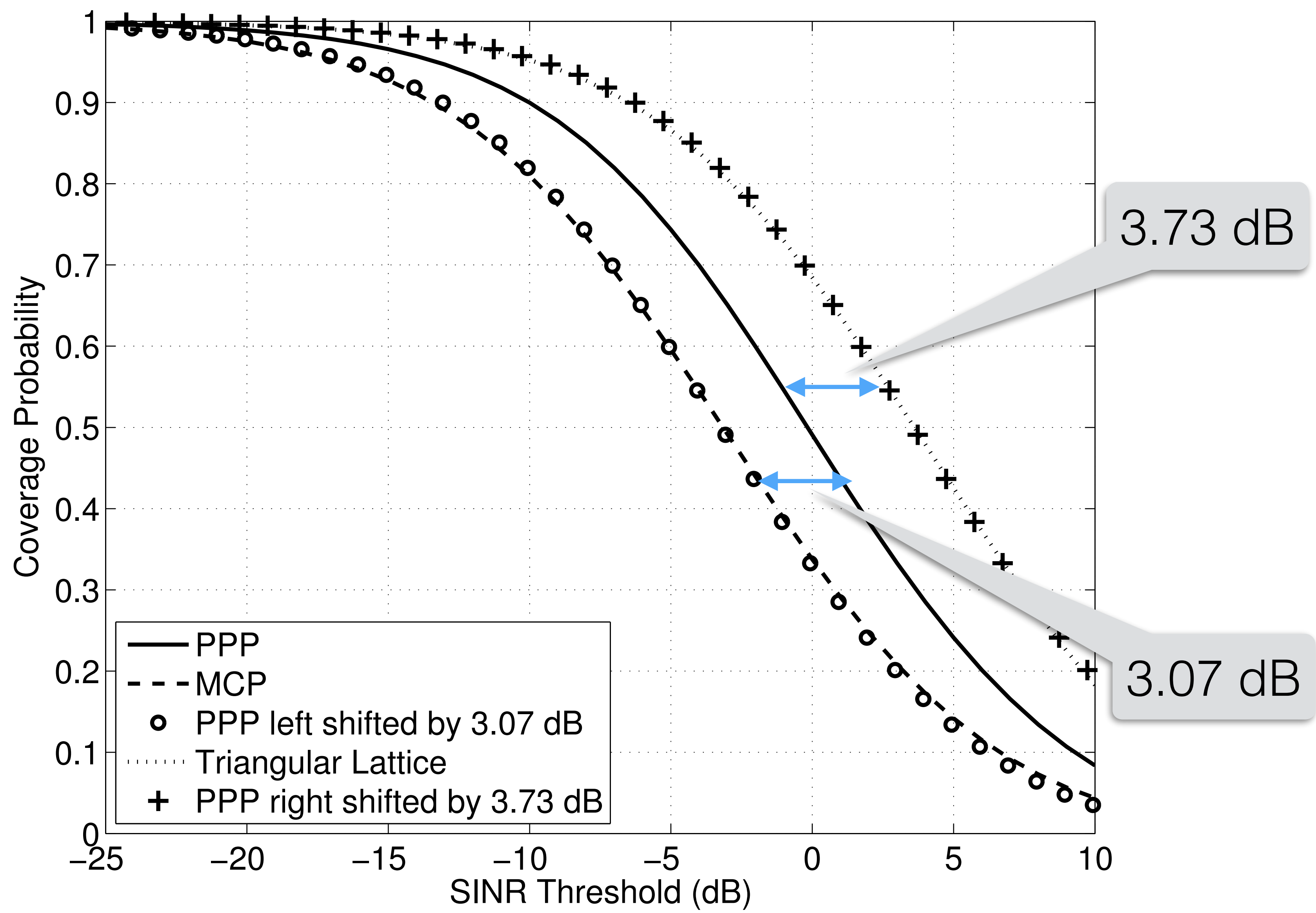
[1] J. G. Andrews, F. Baccelli, and R. K. Ganti. ***A tractable approach to coverage and rate in cellular networks***. *IEEE Trans. on Communications*, 59(11):3122--3134, November 2011

SIR CCDF for other spatial distributions?

- Ginibre point process $\int_0^\infty e^{-v}\Big[\prod_{j=0}^{\infty}\frac{1}{j!}\int_v^\infty \frac{s^j e^{-s}}{1+\theta(v/s)^{\alpha/2}}\mathrm{d}s\Big]\Big[\sum_{i=0}^{\infty} v^i \Big(\int_v^\infty \frac{s^i e^{-s}}{1+\theta(v/s)^{\alpha/2}}\mathrm{d}s\Big)^{-1}\Big]\mathrm{d}v$
- Other point process

Nothing known!

Observation (Martin)



$$P(T) \approx P_{\mathrm{PPP}}(\theta T)$$

Horizontal shift of PPP CCDF (in dB scale)

Horizontal gap

$$G(p) = \frac{F^{-1}(p)}{F_{\mathrm{PPP}}^{-1}(p)}, \quad p \in (0,1)$$

In particular, we focus on G(0) and G(∞)

$$P(T) \approx P_{\mathrm{PPP}}(G(0)T)$$

G(0) and G(∞) depend on the SIR asymptotes at 0 and ∞

Head behaviour

$$P_c(T) = \mathbb{P}\left(\frac{hr^{-\alpha}}{I} > T\right)$$

$$P_c(T) = \mathbb{E}\left[\exp(-Tr^{\alpha}I)\right]$$

$$P_c(T) \sim 1 - T\mathbb{E}\left[\frac{I}{r^{-\alpha}}\right]$$

$$P_c(T) \sim 1 - T\mathrm{MISR}$$

$$G(0) = \frac{\mathrm{MISR}}{\mathrm{MISR}_{\mathrm{PPP}}}$$

Mean interference to signal ratio

Mean interference-to-signal ratio (PPP)

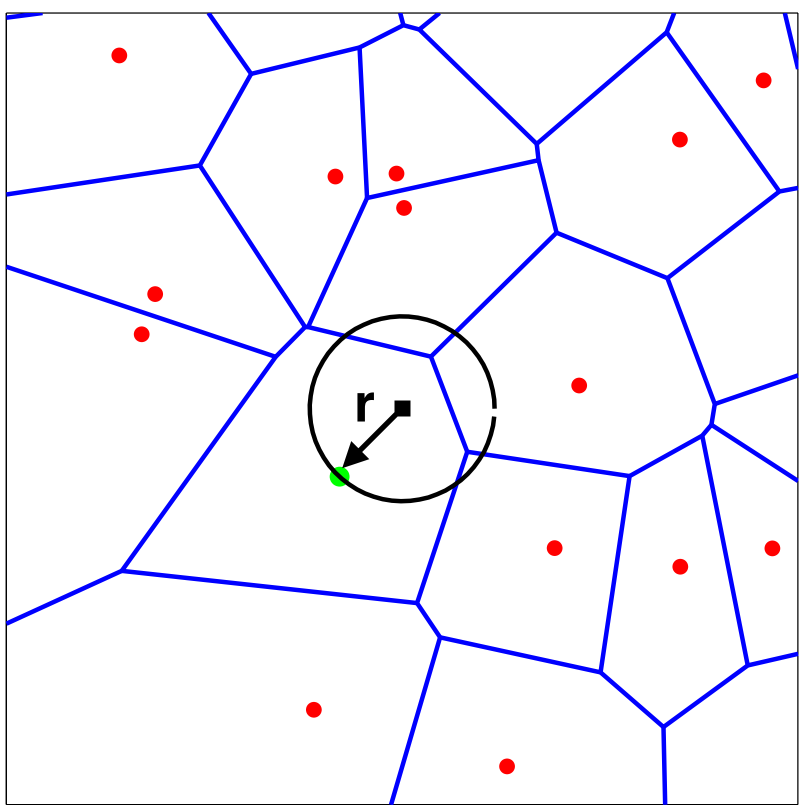
$$\mathrm{MISR}_{\mathrm{PPP}} = \mathbb{E}\left[\frac{I}{r^{-\alpha}}\right] = \mathbb{E}\left[r^{\alpha} \sum_{y \in \Phi \setminus \{y_0\}} h_y \|y\|^{-\alpha}\right]$$

$$= \mathbb{E}_r\left[r^{\alpha} \mathbb{E}\left[\sum_{y \in \Phi \setminus \{y_0\}} h_y \|y\|^{-\alpha}\right]\right]$$

$$= \mathbb{E}_r\left[r^{\alpha} \int_{B(o,r)^c} \|y\|^{-\alpha} dy\right]$$

$$= \frac{2}{\alpha - 2}$$

Base stations: big dots. Mobile users: little dots.



Signal power dominates for higher path loss and vice versa

Relative distance process

CCDF of SIR: $\mathbb{E}\left[F\left(\frac{I}{r^{-\alpha}}\right)\right] = \mathbb{E}\left[F\left(\sum_{y\in\Phi\setminus\{y_0\}} h_y \left(\frac{\|y_o\|}{\|y\|}\right)^{-\alpha}\right)\right]$

$$\mathcal{R} \triangleq \{x \in \Phi \setminus \{y_o\} : \|y_o\|/\|y\|\} \subset (0,1)$$

- Not a stationary process
- Not a finite point process

RDP of PPP(λ)

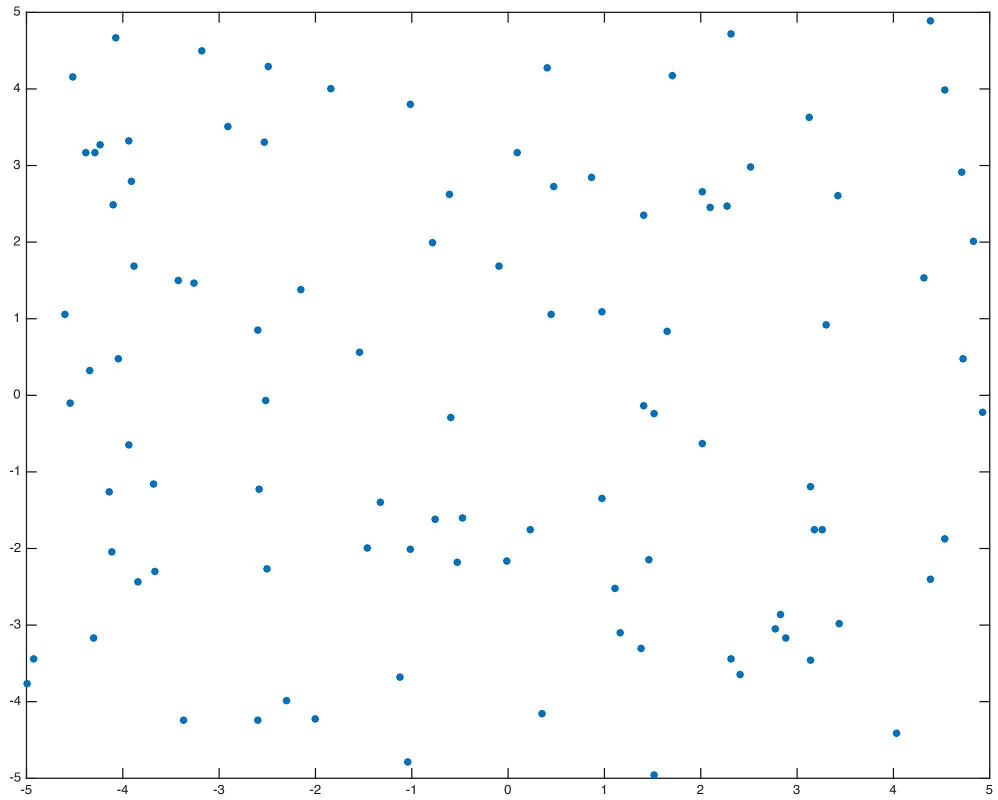
Product densities

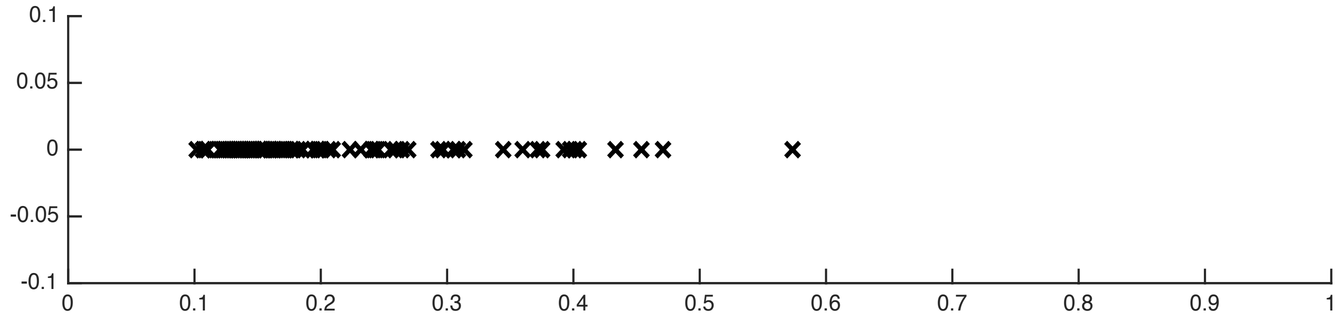
$$\rho_{\mathcal{R}}^{(n)}(t_1, t_2, \ldots, t_n) = n! 2^n \prod_{i=1}^{n} t_i^{-3}$$

Intensity

$$\mathbf{E}[\mathcal{R}([r,1])] = \Lambda([r,1])$$
$$= r^{-2} - 1$$

Complete characterisation





RDP of a general PP

Theorem. For an RDP generated by a stationary PP

$$\rho_{\mathcal{R}}^{(n)}(t_1, t_2, \ldots, t_n) = \beta_n(t_1, ..., t_n) n! 2^n \prod_{i=1}^{n} t_i^{-3}$$

Proof.

Joint pdf of $\frac{\|y_0\|}{\|y\|}, \quad y \in \Phi \setminus y_o$

Get rid of y_o

$$f(y_o, \Phi \setminus y_o) = \sum_{x \in \Phi} f(x, \Phi \setminus x) \mathbf{1}(\Phi(B(o, \|x\|) = 0))$$

MISR of general PP

Theorem. The MISR of a motion-invariant point process is

$$\mathrm{MISR} = 2\int_0^1 t^{\alpha-3}\beta_1(t)dt$$

Proof. $\mathrm{MISR} = E\sum_{y\in\mathcal{R}} y^\alpha = \int_0^1 t^\alpha \rho_{\mathcal{R}}^{(1)}(t)dt$

Dependence on path loss exponent

$$\mathrm{MISR}(\alpha) \sim \frac{2}{\alpha-2}\beta_1(1), \quad \alpha\to\infty$$

Expected fading-to-interference ratio (EFIR)

$$\mathrm{EFIR} \triangleq \left(\lambda \pi \mathbb{E}^{!o} \left[\left(\frac{h}{I_\infty} \right)^{2/\alpha} \right] \right)^{\alpha/2}$$

where $I_\infty = \sum_{x \in \Phi} h_x \|x\|^{-\alpha}$

Examples

PPP

$$\mathrm{EFIR}_{\mathrm{PPP}} = (\mathrm{sinc}(2/\alpha))^{\alpha/2}$$

Square lattice

$$\frac{(\pi\Gamma(1+2/\alpha))^{\alpha/2}}{Z(\alpha)} \leq \mathrm{EFIR}_{\mathrm{lat}} \leq \left(\frac{\pi}{\mathrm{sinc}(2/\alpha)}\right)^{\alpha/2} \frac{1}{Z(\alpha)}$$

Is EFIR independent of density?

Tail behaviour

Theorem. For a stationary BS point process,

$$P_c(T) \sim \left(\frac{T}{\mathrm{EFIR}}\right)^{-2/\alpha}, \quad T \to \infty$$

Remarks:

- SIR is a heavy tailed distribution
- Sanity check: PPP $\quad P_c(T) = \frac{1}{1 + T^{2/\alpha} \int_{T^{-2/\alpha}}^{\infty} \frac{1}{1+u^{\alpha/2}} du} \sim \frac{T^{-2/\alpha}}{\int_0^{\infty} \frac{1}{1+u^{\alpha/2}} du}$
- Similar results for Ginibre process [Miyoshi 14]

Relation to max SIR

Lemma. The tail of the max SIR distribution coincides with nearest neighbour connectivity, *i.e.,*

$$\left(\frac{T}{\mathrm{EFIR}}\right)^{-2/\alpha} \sim P_c(T) \sim \mathbb{P}(\max_{x\in\Phi} \mathrm{SIR}(x) > T)$$

For large SIR threshold, signal fade does not matter

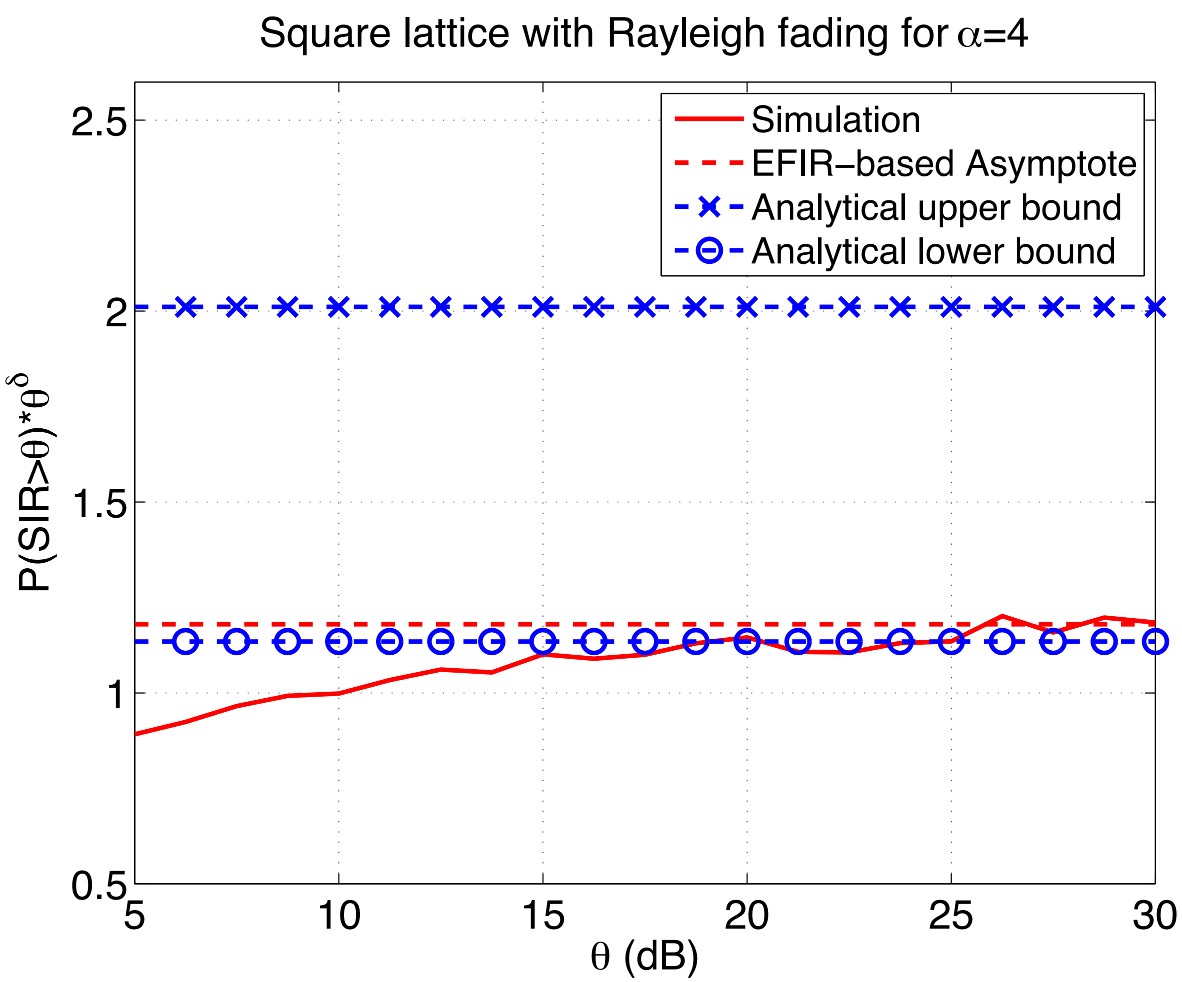
Forget interference

The received signal strength: $S = hr^{-\alpha}$

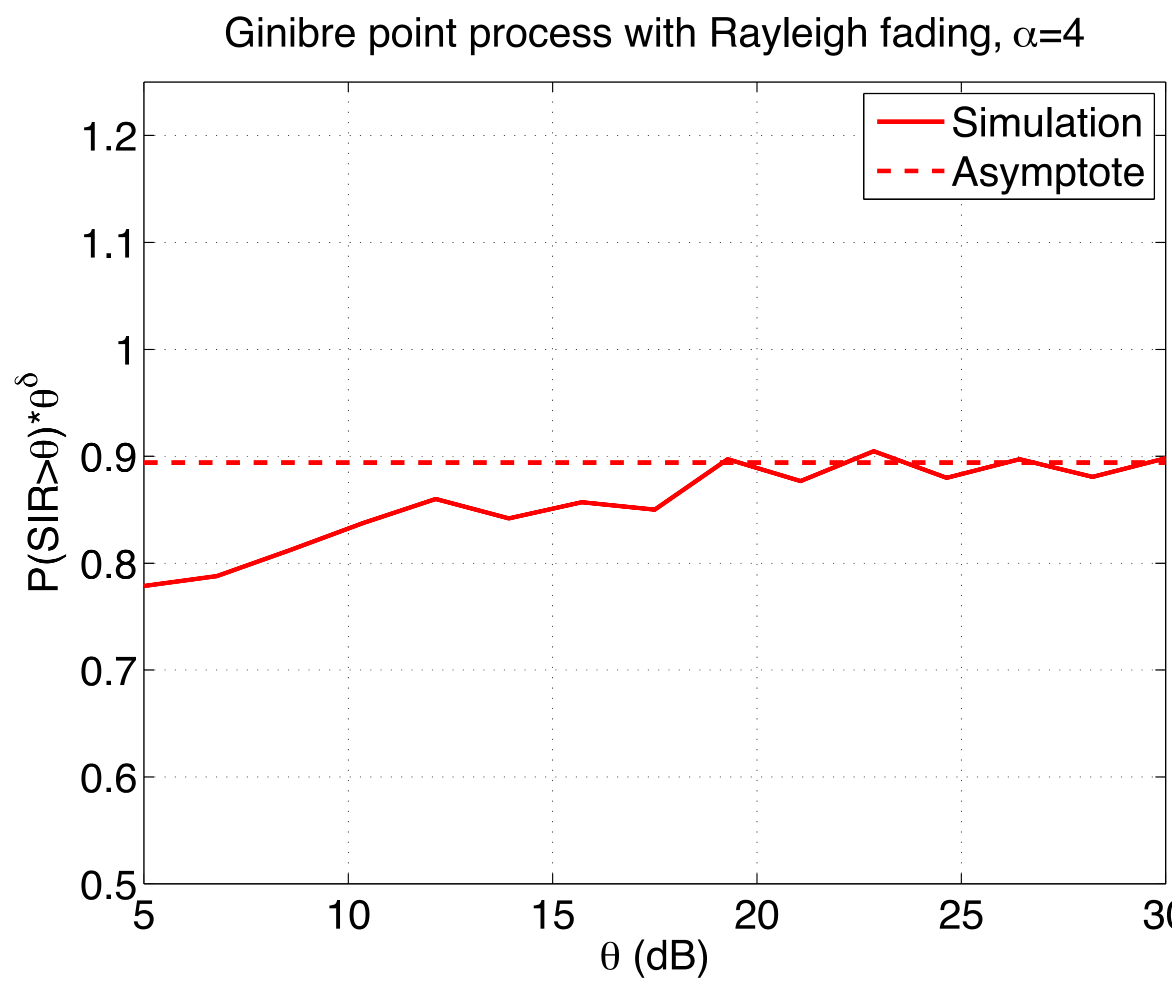
$$\mathbb{P}(S > T) \sim \lambda\pi\Gamma(1 + 2/\alpha)T^{-2/\alpha}, \quad T \to \infty$$

$$\mathbb{P}(S < T) \sim T\mathbb{E}[r^{\alpha}], \quad T \to 0$$

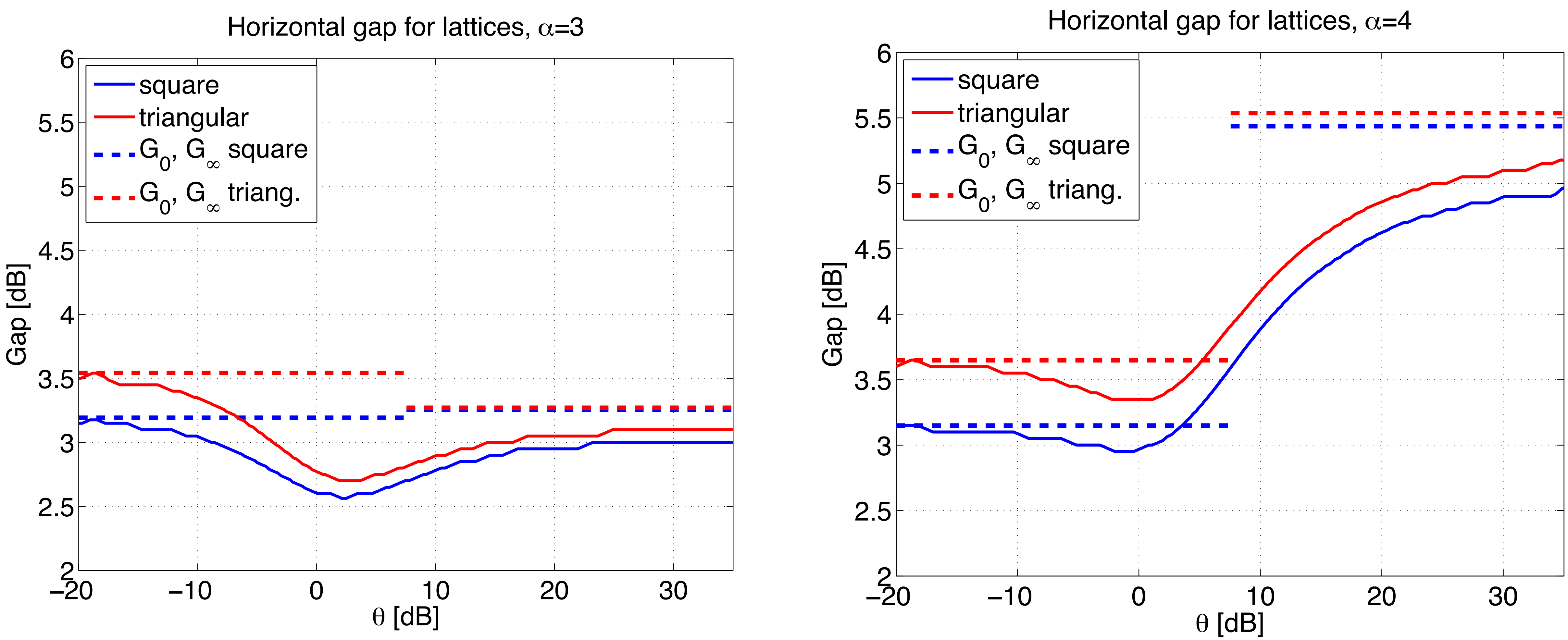
At both ends of the SIR distribution, the interference affects only the pre-constant



$$\sqrt{\mathrm{EFIR}} = 1.19$$



$$\sqrt{\mathrm{EFIR}} = 0.89$$

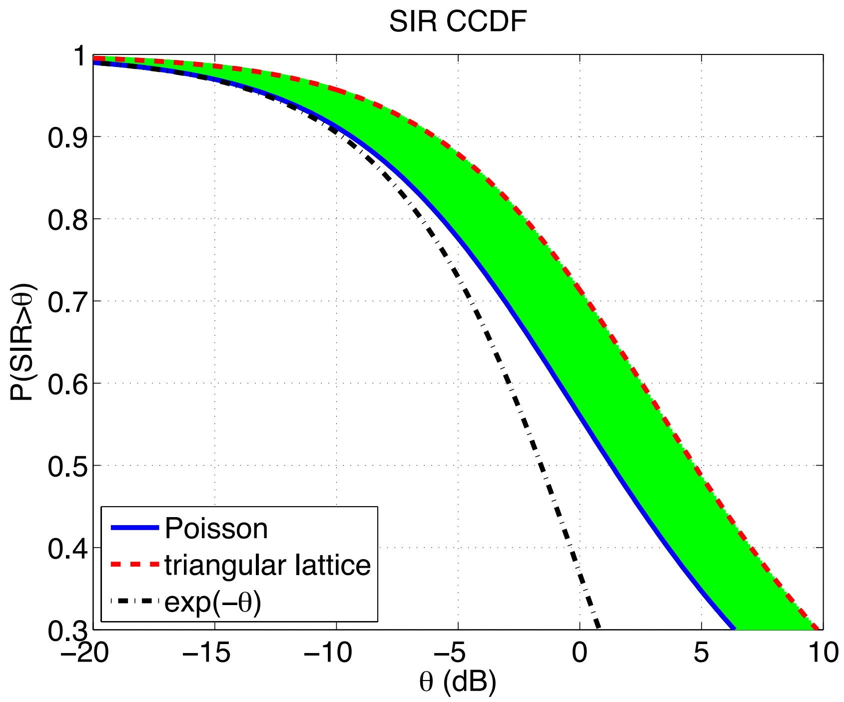

Horizontal gap for lattices, α=3
Horizontal gap for lattices, α=4
square
triangular
G0, G∞ square
G0, G∞ triang.
Gap [dB]
θ [dB]
6
5.5
5
4.5
4
3.5
3
2.5
2
–20
–10
0
10
20
30

Conclusion

- SIR distribution asymptotes are characterised by MSIR and EFIR.

- RDP process captures the relevant information for computing SIR distribution. Easier to analyse??

- SIR CDF for non-PPP BSs is a horizontal shift of the PPP CDF



Just use PPP!