Efficient Recovery from Multiple Erasures by Accessing Small Number of Disks in Distributed Data Storage

Ganesh R. Kini and Balaji S.B.
Codes and Signal Design Lab
Advisor: Prof Vijay Kumar

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Students' Seminar Series
Department of ECE
Indian Institute of Science

Outline

Codes with Sequential Local Recovery

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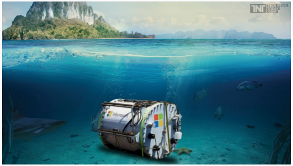
Distributed Storage



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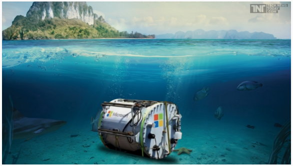

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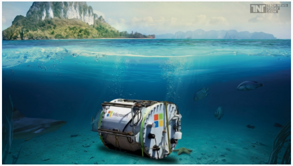

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- Requirements:
  - High reliability i.e. protection from data loss due to disk failures
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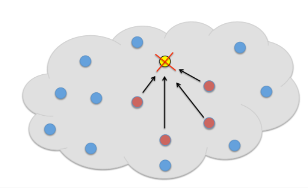
  - Low storage overhead
    i.e. want high-rate codes
  - Efficient repair of a disk when it fails
    want to contact very few surviving nodes



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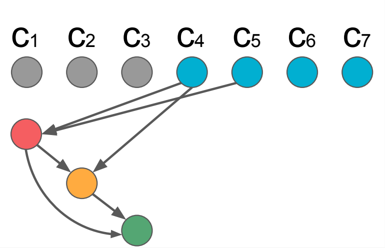
Property:

$c_1 = f_1(c_4, c_5)$, $c_2 = f_2(c_1, c_4)$, $c_3 = f_3(c_1, c_2)$

So that, can recover the lost symbols in the sequence $c_1 - -c_2 - -c_3$.



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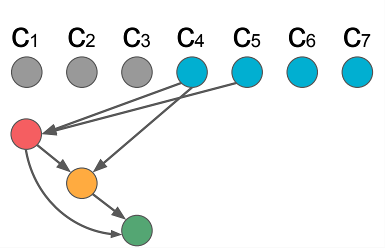
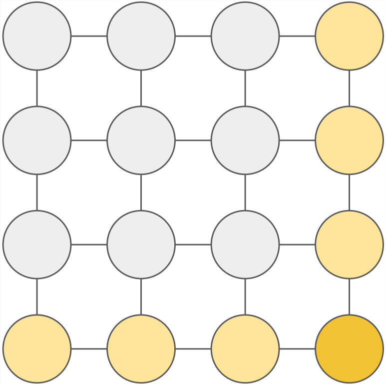


Questions: What is the highest "rate" achievable by such codes? How to design such rate-optimal codes with low blocklength, low field-size?

Outline

A Simple Code with Sequential Recovery: 2D Product Code



- $(n = 16, k = 9, r = 3, t = 3)_{\text{seq}}$ code
- Rate of the code for general r is $\frac{k}{n} = \frac{r^2}{(r+1)^2}$
- Every row is a codeword of SPC code, every column is a codeword of SPC code
- Parity is the sum of r symbols

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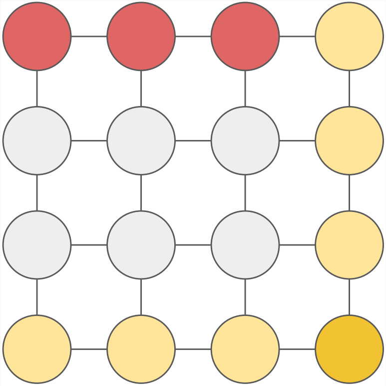

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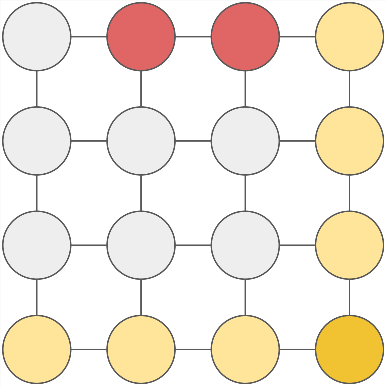

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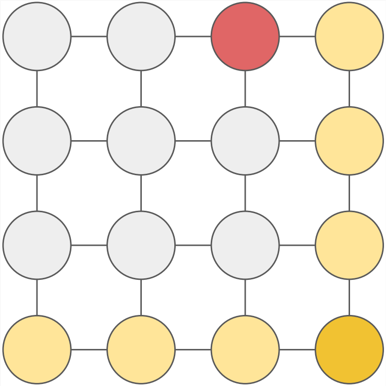

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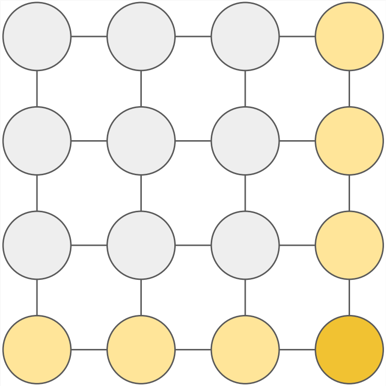

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In any sequence?

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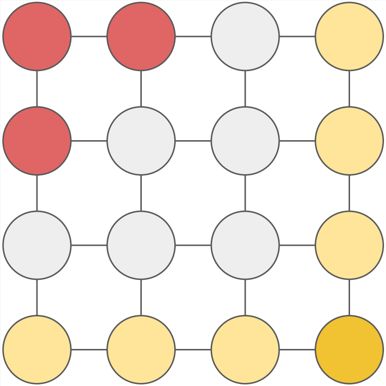

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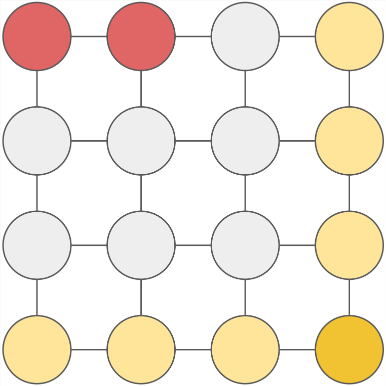

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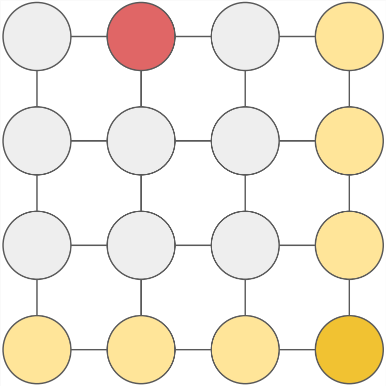

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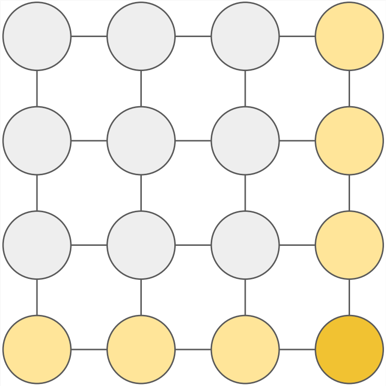

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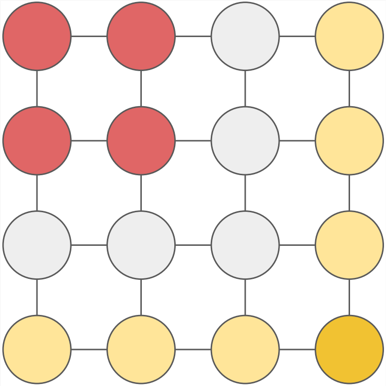

- Hence can correct any 3-erasure-pattern
- But some 4-erasure-patterns are uncorrectable

Definition of Code with Sequential Local Recovery(Sequential LRC)

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Code with Sequential Local Recovery

An $[n, k]$ code is said to be a locally recoverable code with sequential recovery from t erasures, if for any set of $s \leq t$ erasures, there is an s-step sequential recovery process, in which at each step, a single erased symbol is recovered by accessing at most r other code symbols.

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We will formally refer to this class of codes as $(n, k, r, t)_{seq}$ codes.

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Question of Code-Rate

Given locality parameter r and erasure correctability parameter t, what is the maximum achievable code-rate?

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Theorem

Rate Bound[1] *: Let $\mathcal{C}$ be an $(n, k, r, t)_{seq}$ code over a field $\mathbb{F}_q$. Let $r \geq 3$. Then*

$$\frac{k}{n} \leq \frac{r^{\frac{t}{2}}}{r^{\frac{t}{2}} + 2\sum_{i=0}^{\frac{t}{2}-1} r^i} \quad \text{for even } t, \qquad (1)$$

$$\frac{k}{n} \leq \frac{r^s}{r^s + 2\sum_{i=1}^{s-1} r^i + 1} \quad \text{for odd } t, \qquad (2)$$

where $s = \frac{t+1}{2}$.

[1] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures, 2017. [Online]. Available: http://arxiv.org/abs/1611.08561

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Proof.

We investigate the structure of the parity check matrix

Parity-Check Matrix

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- Let $H_1 = \begin{bmatrix} \underline{c_1} \\ \underline{c_2} \\ \vdots \\ \underline{c_m} \end{bmatrix}$
- H_1 is a parity-check matrix of an $(n, n-m, r, t)_{seq}$ code

Parity Check Matrix of a Linear Code

Suppose an n-length code has code-symbols $c_1, ..., c_n$.
The rows of a parity check matrix of the code are nothing but the linear equations that the code-symbols satisfy.

Suppose $[a_1, ..., a_n]$ is one row, then $\sum_{i=1}^{n} a_i c_i = 0$

If the dimension of the code is k, then the parity check matrices have rank $n - k$.

P-C Matrix for Sequentially Correcting t Erasures Locally

We'll now see the case of even t

P-C Matrix for Sequentially Correcting t Erasures Locally

Start with any (n, k, r, t) code, consider the matrix H_1 for it, with row and column permutations it looks like this:

$$H_1 = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\
\cline{1-8}
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\
\cline{1-8}
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 & \\
\cline{1-8}
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 & \\
\cline{1-8}
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & E \\
\cline{1-8}
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 & \\
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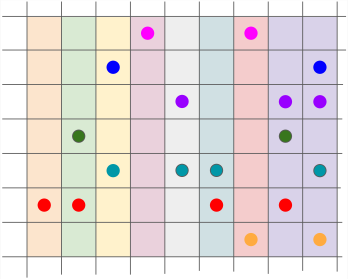
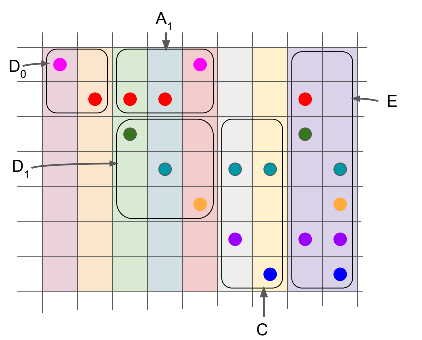
0 & 0 & 0 & 0 & \dots & D_{\frac{t}{2}-2} & A_{\frac{t}{2}-1} & 0 & \\
\cline{1-8}
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & & \\
\cline{1-7}
0 & 0 & 0 & 0 & \dots & 0 & 0 & C &
\end{array}\right]$$

P-C Matrix: Row and Column Permutation

Take H_1 matrix of any $(n, k, r, t)_{seq}$ code. Permute rows and columns to get the staircase form:





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- C: columns have weight exactly 2

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- A_i: columns have weight at least 1
- D_i: rows have weight at least 1 and columns have weight at most 1
- C: columns have weight exactly 2
- E: columns have weight at least 3

P-C Matrix: Structure

Claim

A_i's are matrices with each column having weight 1 and D_i's are matrices with each row and each column having weight 1.

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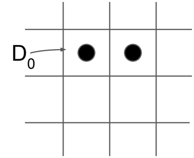
If D_0 has a row with at least 2 non-zero entries, then 2 columns become linearly dependent, a contradiction to $d_{min}(\mathcal{C}) \geq t+1$.



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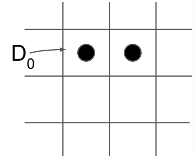

Therefore, with column permutation, D_0 is diagonal.

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A_1: columns have weight exactly 1

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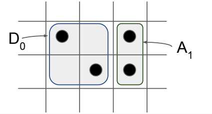

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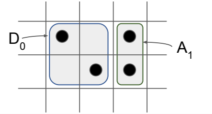

3 columns linearly dependent.

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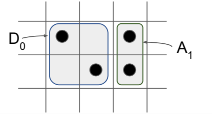

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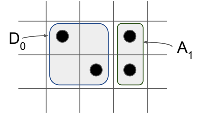
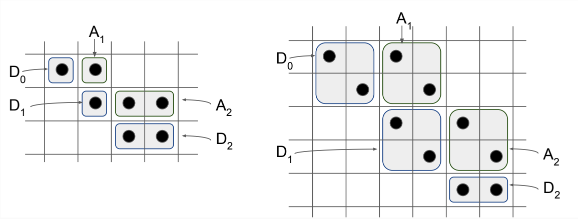


3 columns linearly dependent.

D_2: rows have weight exactly 1



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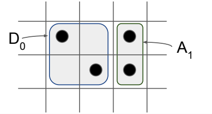

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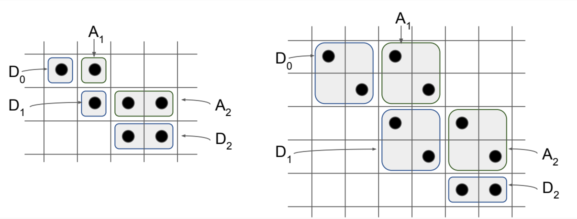

Upto 6 columns linearly dependent.

P-C Matrix: Structure

For some $0 \leq i \leq \frac{t}{2} - 1$, upto $2(i+1) \leq t$ columns become linearly dependent, which is a contradiction to $d_{min}(\mathcal{C}) \geq t + 1$.

Thus, the Claim is true. i.e.
A_i's are matrices with each column having weight 1 and D_i's are matrices with each row and each column having weight 1.

Therefore, D_i's are diagonal (identity, after scaling) matrices with number of rows ρ_i and number of columns a_i equal.

$$\rho_i = a_i$$

Let's recall...

$$
m \quad
\begin{array}{c}
\\
a_0 \\ a_1 \\ \\ \\ \\ \\ \\ a_{t/2-1} \\ p
\end{array}
\quad
H_1 =
\begin{array}{c}
\begin{array}{cccccccc}
a_0 & a_1 & & & & & a_{t/2-1} & a_{t/2}
\end{array}
\\
\left[
\begin{array}{cccccccc|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 & \\
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & E \\
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 & \\
0 & 0 & 0 & 0 & \dots & D_{\frac{t}{2}-2} & A_{\frac{t}{2}-1} & 0 & \\
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & & \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & C &
\end{array}
\right]
\\
n
\end{array}
$$

Now we count...

Equating sum of row-weights and sum of column-weights of A_i:

$$
H_1 = \begin{array}{cc}
 & \begin{array}{ccccccccc} \quad & \quad & \quad & a_3 & \quad & \quad & \quad & \quad & \quad \end{array} \\
\begin{array}{c} \\ \\ a_2 \\ \\ \\ \\ \\ \\ \\ \end{array} &
\left[\begin{array}{c|c|c|c|c|c|c|c|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\
\hline
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\
\hline
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 & \\
\hline
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 & \\
\hline
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & E \\
\hline
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 & \\
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\hline
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & & \\
\cline{1-7}
0 & 0 & 0 & 0 & \dots & 0 & 0 & C &
\end{array}\right]
\end{array}
$$

$$a_{i-1} r \geq a_i$$

Now we count...

Equating sum of row-weights and sum of column-weights of C:

$$H_1 = \begin{array}{ll}
 & \begin{array}{cccccccccc} & & & & & \phantom{A_{\frac{t}{2}-2}} & a_{t/2-1} & a_{t/2} & \end{array} \\
\begin{array}{r} \\ \\ \\ \\ \\ \\ \\ a_{t/2-1} \\ p \end{array} & \left[\begin{array}{cccccccc|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 & \\
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\end{array}\right]
\end{array}$$

$$2a_{\frac{t}{2}} \leq (a_{\frac{t}{2}-1} + p)(r+1) - a_{\frac{t}{2}-1}$$

Now we count...

Equating sum of row-weights and sum of column-weights of H_1:

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D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\ \hline
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\ \hline
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\end{array}\right]
$$

$$
m(r+1) \geq a_0 + 2\left(\sum_{i=1}^{\frac{t}{2}} a_i\right) + 3\left(n - \sum_{i=0}^{\frac{t}{2}} a_i\right)
$$

The Inequalities

We now have the following set of inequalities:

$$a_{i-1}r \geq a_i \tag{3}$$

$$2a_{\frac{t}{2}} \leq (a_{\frac{t}{2}-1} + p)(r+1) - a_{\frac{t}{2}-1} \tag{4}$$

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Now we obtain a lower bound on m

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Proof for odd t proceeds along similar lines

Conditions for Equality

- $a_i = \frac{2nr^i}{r^{\frac{t}{2}} + 2\sum_{j=0}^{\frac{t}{2}-1} r^j}$, for $0 \leq i \leq \frac{t}{2} - 1$,
- $a_{\frac{t}{2}} = \frac{nr^{\frac{t}{2}}}{r^{\frac{t}{2}} + 2\sum_{j=0}^{\frac{t}{2}-1} r^j}$,
- $p = 0$
- Note that $\sum_{i=0}^{\frac{t}{2}} a_i = n$, therefore E is an empty matrix.

The parity-check matrix then is

$$H_1 = \left[\begin{array}{c|c|c|c|c|c|c|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 \\ \hline
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & \dots & D_{\frac{t}{2}-2} & A_{\frac{t}{2}-1} & 0 \\ \hline
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & C
\end{array}\right]$$

Outline

1. Codes with Sequential Local Recovery
   - Introduction
   - An Example: 2D Product Code
   - Upper Bound on Code Rate
   - A Rate-Optimal Binary Code Construction
2. Codes with Availability
   - A Greedy Algorithm for Rate-Bound

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- An iterative procedure for constructing a graph $G_{\frac{t}{2}-1}$ starting from a regular graph G_0
- Add nodes to the graph in every step in a layer-by-layer fashion, each time maintaining the girth of graph to be at least $t+1$

Graph Construction

$t = 4, r = 3$ construction

Pick G_0, r-regular, girth $\geq (t+1)$,
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$3-$ regular graph
with $u_0 = 10$ nodes
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$\downarrow$

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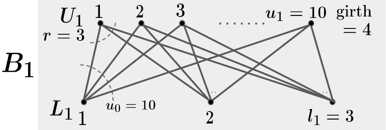

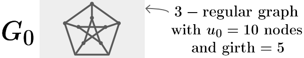

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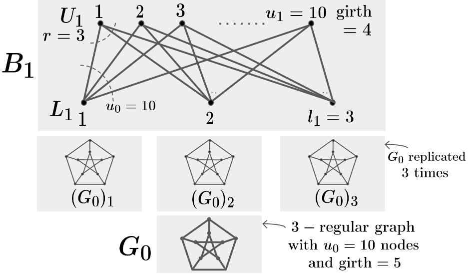

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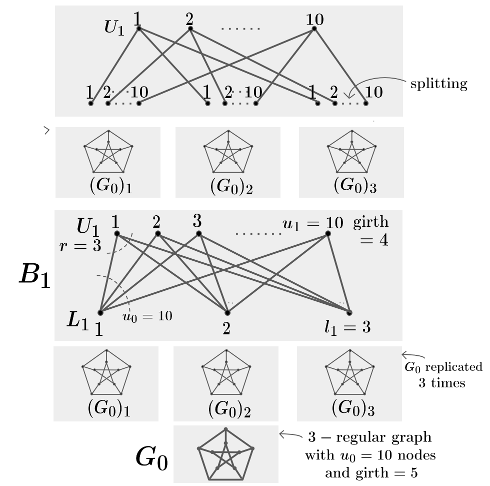

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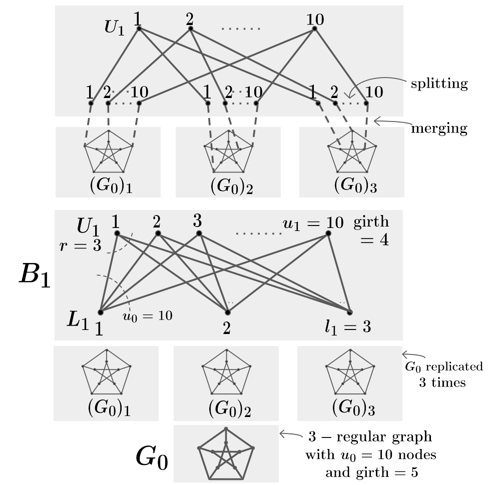

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The resulting graph is G_i; can be verified that G_i has girth $\geq t+1$, the nodes $U_i \subset V(G_i)$ now form the upper layer of the graph G_i and these are the nodes in G_i that participate in the next iterative step

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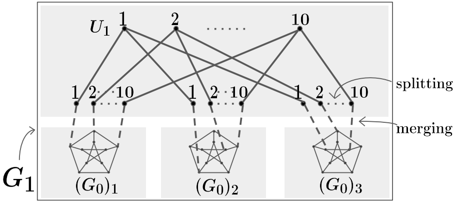

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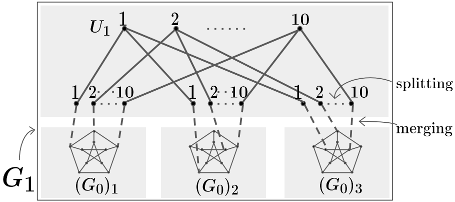
is $i = \frac{t}{2} - 1$?

$t = 4, r = 3$ construction



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Merge the u_{i-1}, degree-1 nodes obtained from splitting a single node of L_i with the u_{i-1} nodes lying in U_{i-1} and corresponding to a single copy of G_{i-1}

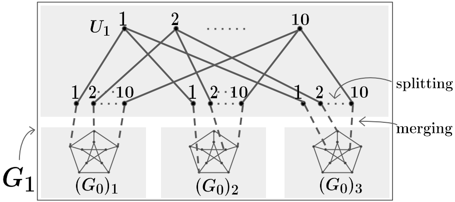
The resulting graph is G_i; can be verified that G_i has girth $\geq t+1$, the nodes $U_i \subset V(G_i)$ now form the upper layer of the graph G_i and these are the nodes in G_i that participate in the next iterative step

is $i = \frac{t}{2} - 1$?

NO: $i = i + 1$

YES: stop

$t = 4, r = 3$ construction



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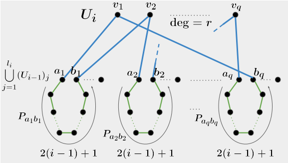

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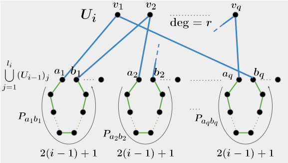

$$\begin{aligned}
\text{Length of any cycle } &\geq 2q + q(2(i-1)+1) \qquad (9)\\
&\geq g_{B_i} + \frac{g_{B_i}}{2}(2(i-1)+1)\\
&\geq g_{B_i}(i+\frac{1}{2}) \geq \frac{t+1}{i+\frac{1}{2}}(i+\frac{1}{2}) = t+1
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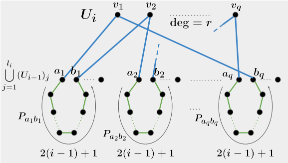

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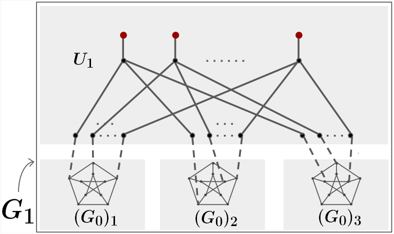
Hence for every i, girth of G_i is at least $t+1$

Code Defined on the Graph: Tanner Graph



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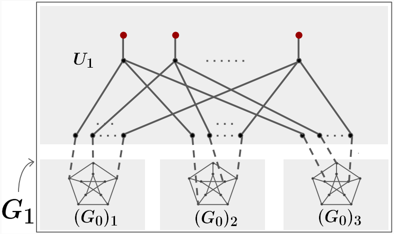

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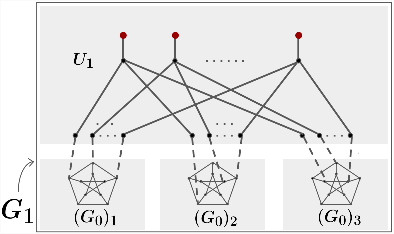

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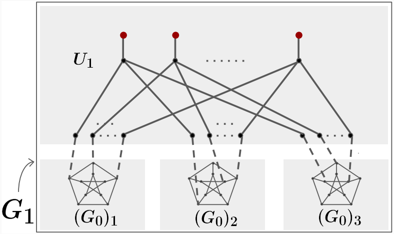

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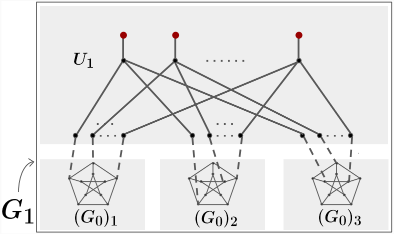

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Erasure Correction

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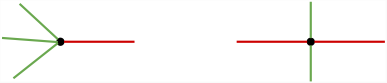

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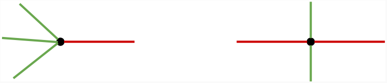

Hence, for correcting multiple erasures, at every step there should exist at least one parity check, with exactly one erased symbol.
When do we run into trouble?

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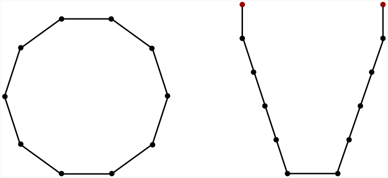

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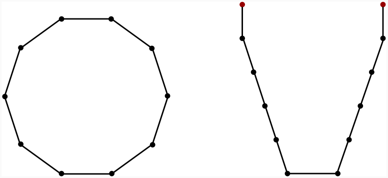

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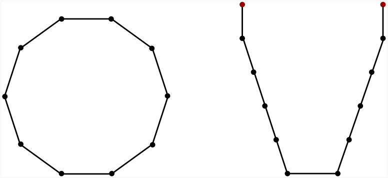


- The graph does not have cycles of length less than $t+1$. Therefore the first case does not arise
- Any two dummy nodes are separated by a path of length at least $t+1$. Hence the second case is also avoided

Rate-Optimality of the construction

Upon counting the number of code symbols and number of parity checks, one can see that this construction yields rate-optimal codes for any even t and any $r \geq 3$.

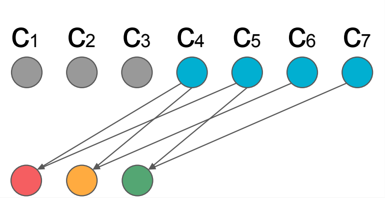
Codes with Availability

Outline

Availability

Same setting as earlier

Desirable:
$c_1 = f_1(c_4, c_5)$, $c_2 = f_2(c_4, c_6)$,
$c_3 = f_3(c_5, c_7)$
So that, can recover the lost symbols in any order.



An additional property called "majority-logic decodability".

Questions:
What is the highest "rate" achievable by such codes? Highest "minimum-distance"? How to design such rate-optimal codes with low blocklength, low field-size?

Definition: Strict Availability via Orthogonal Parity Checks

$$\underbrace{H}_{(m\times n)} = \underbrace{\begin{bmatrix} 1 & 1 & & 1 & & & \\ & 1 & 1 & & 1 & & \\ & & 1 & 1 & & 1 & \\ & & & 1 & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 1 \end{bmatrix}}_{\Updownarrow \ w_H(\text{each column})=t \ \Updownarrow} \Leftrightarrow w_H(\text{each row}) = (r+1)$$

- Let $S_j^{(i)}$ be the support of the jth row having a 1 in column i
- Then

$$S_j^{(i)} \cap S_l^{(i)} = \{i\}, \quad j = 1, 2, \cdots, t, \quad j \neq l.$$

- t orthogonal parity checks per code symbol
- Terminology: $(n, k, r, t)_{\text{sa}}$ codes.

Outline

Rate-Bound for Strict Availability for t $= 3$

Theorem

Rate-Bound[1] *Let $\mathcal{C}$ be an $(n, k, r, 3)_{sa}$ code over the field $\mathbb{F}_q$ having connected Tanner graph, then its rate is upper bounded by the following expression:*

$$\frac{k}{n} \leq 1 - \frac{3(1 + L_1 + L_2)}{(r+1)(3 + L_1 + 2L_2)}, \tag{10}$$

$$\text{where: } m = \frac{3n}{r+1}, \quad L_1 = \left\lceil \frac{(2r-1)m}{3(r+2)} - \frac{1}{r+2} - 1 \right\rceil,$$
$$L_2 = \left\lfloor \frac{m - 3 - L_1}{2} \right\rfloor,$$

[1] S. B. Balaji and P. Vijay Kumar, “Bounds on Codes with Locality and Availability”,2017. [Online] .Available: https://arxiv.org/abs/1611.00159

Greedy Algorithm : Step 1

$$H = \begin{bmatrix} 1 & 1 & & 1 & & & \\ & 1 & 1 & & 1 & & \\ & & 1 & 1 & & 1 & \\ & & & 1 & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 1 \end{bmatrix}$$

$S = \{1\}$,

$$P = \begin{bmatrix} 1 & 1 & & 1 & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 1 & & & & 1 & 1 & \\ & & & & & & \\ 1 & & 1 & & & & 1 \end{bmatrix}$$

Greedy Algorithm: Step 2

$$H = \left[\begin{array}{c|c|c|c|c|c|c}
1 & 1 & & 1 & & & \\ \hline
 & 1 & 1 & & 1 & & \\ \hline
 & & 1 & 1 & & 1 & \\ \hline
 & & & 1 & 1 & & 1 \\ \hline
1 & & & & 1 & 1 & \\ \hline
 & 1 & & & & 1 & 1 \\ \hline
1 & & 1 & & & & 1
\end{array}\right]$$

$S = \{1, 2\}$,

$$P = \left[\begin{array}{c|c|c|c|c|c|c}
1 & 1 & & 1 & & & \\ \hline
 & 1 & 1 & & 1 & & \\ \hline
 & & & & & & \\ \hline
 & & & & & & \\ \hline
1 & & & & 1 & 1 & \\ \hline
 & 1 & & & & 1 & 1 \\ \hline
1 & & 1 & & & & 1
\end{array}\right]$$

Greedy Algorithm: Step 3

$$H = \begin{bmatrix} 1 & 1 & & 1 & & & \\ & 1 & 1 & & 1 & & \\ & & 1 & 1 & & 1 & \\ & & & 1 & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 1 \end{bmatrix}$$

$S = \{1, 2, 5\}$,

$$P = \begin{bmatrix} 1 & 1 & & 1 & & & \\ & 1 & 1 & & 1 & & \\ & & & & & & \\ & & & 1 & 1 & & 1 \\ 1 & & & & 1 & 1 & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & 1 \end{bmatrix}$$

Greedy Algorithm: Step 4

$$H = \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & & 1 & & & \\ \hline & 1 & 1 & & 1 & & \\ \hline & & 1 & 1 & & 1 & \\ \hline & & & 1 & 1 & & 1 \\ \hline 1 & & & & 1 & 1 & \\ \hline & 1 & & & & 1 & 1 \\ \hline 1 & & 1 & & & & 1 \end{array}\right]$$

$S = \{1, 2, 5, 6\}$, Hence $P = H$ and $k \leq n - |S| = 3$

$$P = \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & & 1 & & & \\ \hline & 1 & 1 & & 1 & & \\ \hline & & 1 & 1 & & 1 & \\ \hline & & & 1 & 1 & & 1 \\ \hline 1 & & & & 1 & 1 & \\ \hline & 1 & & & & 1 & 1 \\ \hline 1 & & 1 & & & & 1 \end{array}\right]$$

Greedy Algorithm

1. Let $S = \emptyset, P = \emptyset$.
2. Step 1: Pick an arbitrary number σ_1 from $[n]$ and set $S = \{\sigma_1\}$ and $P = \{\underline{c} \in \text{Rows}(H) : \sigma_1 \in \text{Support}(\underline{c})\}$.
3. Step i, $i \geq 2$: Choose a number $\sigma_i \in [n] - S$ such that
   $\sigma_i = \text{argmax}_{\{j \in [n] - S\}} |D_j| \times I(|D_j| \leq 2)$
   where $D_j = \{\underline{c} \in P : j \in \text{Support}(\underline{c})\}$.
   Now $S = S \cup \{\sigma_i\}$ and
   $P = P \cup \{\underline{c} \in \text{Rows}(H) - P : \sigma_i \in \text{Support}(\underline{c})\}$.
4. **Pseudocode for the Greedy Algorithm:** $P = \emptyset, S = \emptyset$, $i = 1$.
   while $|P| < m$
   do Step i
   $i = i + 1$
   end while.
5. It is clear that, $k \leq n - |S|$ at the end of the algorithm.

Backup Slides

Rawat et al. Regular Graph Construction

Any $(r+1)$-regular (bipartite) graph, with girth at least $t+1$ gives the Tanner graph of an $(n, k, r, t)_{seq}$ code.

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Any $(r+1)$-regular (bipartite) graph, with girth at least $t+1$ gives the Tanner graph of an $(n, k, r, t)_{seq}$ code.
Can show that rate is $\frac{r-1}{r+1} + \frac{1}{n}$.
Can show that it meets our bound only when the graph is a Moore graph, which are very rare. Hence sub-optimal for most parameters.

Analysis of Greedy Algorithm

Let $g_i \in \{1, 2\}$ be the number of new codewords added to P at step i. Let s_1^i, s_2^i, s_3^i be the number of weight $1, 2, 3$ columns in the matrix formed by codewords in P respectively.

$$\text{if } g_{i+1} = 2, g_{i+2} = 2 \text{ then}$$
$$s_1^{i+1} = s_1^i + 2r - 1, \;\; s_2^{i+1} = s_2^i + 0, \;\; s_3^{i+1} = s_3^i + 1.$$

$$\text{if } g_{i+1} = 1, g_{i+2} = 2 \text{ then}$$
$$s_1^{i+1} = s_1^i - \phi_i + r + 1, \;\; s_2^{i+1} = s_2^i - \phi_i, \;\; s_3^{i+1} = s_3^i + \phi_i,$$
$$\text{for some } 1 \leq \phi_i \leq r + 1.$$

$$\text{if } g_{i+1} = 2, g_{i+2} = 1 \text{ then}$$
$$s_1^{i+1} = s_1^i + 2r - 1 - 2l_i, \;\; s_2^{i+1} = s_2^i + l_i, \;\; s_3^{i+1} = s_3^i + 1,$$
$$\text{for some } 0 < l_i \leq 2r.$$

Analysis of Greedy Algorithm

Let $S_{uj} = \{i : g_{i+1} = u, g_{i+2} = j, |S| - 1 \geq i \geq 2\}$ and $l_{uj} = |S_{uj}|$ at the end of the algorithm. Now using the global constraints ($s_1 = s_1^{|S|} = 0$, $s_2 = s_2^{|S|} = 0$, $s_3 = s_3^{|S|} = n$ at the end of the algorithm.):

$$
\begin{aligned}
s_2^{|S|} &= \gamma_1 - \sum_{i \in S_{12}} \phi_i + \sum_{i \in S_{21}} I_i - \sum_{i \in S_{11}} J_i + \sum_{i \in S_{11}} \psi_i - (r+1) = 0, \\
m &= \frac{3n}{r+1} = 5 + g_3 + 2(l_{22} + l_{12}) + l_{21} + l_{11}.
\end{aligned}
$$

Analysis of Greedy Algorithm

$$l_{11} + l_{21} \quad \geq \quad \frac{(2r-1)m}{3(r+2)} - \frac{1}{r+2} - 1.$$

1. For $j = 1, 2$, let $L_j = |\{i : g_i = j, |S| \geq i \geq 1\}|$ at the end of the algorithm.
2. $|S| = L_1 + L_2 + 1$ and $m = L_1 + 2L_2 + 3$. Hence $L_1 \geq l_{11} + l_{21}$.

References

S. B. Balaji, G. R. Kini, and P. V. Kumar, “A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures, 2017. [Online].
Available: http://arxiv.org/abs/1611.08561

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Thank you!

Questions?