Channel Coding Rate over a Block Fading Channel in the Finite Blocklength Regime

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Outline

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- Motivation & Setting

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- Final Remarks

Era of Short Packets

- Upcoming wireless systems ought to support novel traffic types (e.g., machine-to-machine)
- Typically short packets are involved (traffic generated by sensors)
- **Stringent** reliability and **latency** requirements
- Need to **refine** classical physical layer **performance metrics** (ergodic capacity, outage capacity etc.)

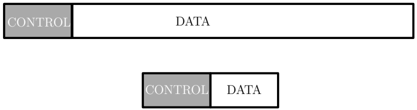

Figure 1: Short packet: Size of metadata and data are comparable

Information Theoretic Metrics

Delay sensitive data over a Wireless Channel

- Earlier: Delay-limited capacity, outage capacity, average capacity etc.
- What's relevant: $R^*(n, \epsilon)$ - maximal coding rate at blocklength n, probability of error ϵ
- But,

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- What's relevant: $R^*(n, \epsilon)$ - maximal coding rate at blocklength n, probability of error ϵ
- But, complexity of computing $R^*(n, \epsilon)$ is prohibitive

A New Benchmark

Delay sensitive data over a Wireless Channel

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- Normal approximation [Polyanskiy-Poor-Verdú, 2010]: For many channels of interest,

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- Question:

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  - $C-$ Channel capacity, $V-$ *Channel dispersion*, $\Phi^{-1}(.)-$ Inverse Gaussian cdf
- Question: Finite blocklength approximations for fading channels?

Channel Model

- Fading takes values in $\{\eta_0, \eta_1, \ldots, \eta_N\}$, $\eta_i \neq 0$

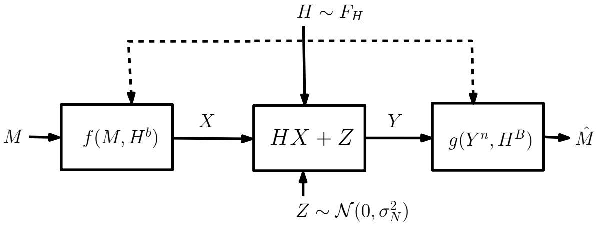

Figure 2: A blockfading channel

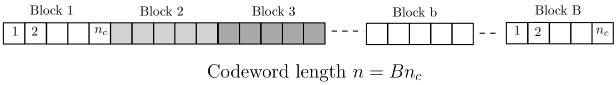

Figure 3: Independent, identically distributed gains across blocks

Code Constraints

- Average probability of error: $0 < \epsilon < \frac{1}{2}$
- Power constraint:
  - Short term ($\mathbf{ST}$) constraint:

$$\sum_{b=1}^{B} \sum_{k=1}^{n_c} X_{[bk]}^2(m, H^b) \leq B n_c \bar{P}, \; \forall m \in [1 : M].$$

- $\mathbf{ST}$ constraint: Corresponds to peak-power limitations of the circuitry

Related Work

- Dispersion of fading channel with CSIR and no CSIT (Polyanskiy and Verdú, ISIT'11)
- Second order coding rate of a DMC with finite states and with non causal CSIT and CSIR (Tomamichel & Tan, T-IT,Aug'14)
- Back off from outage capacity of block fading channel with a single fading block with CSIT and CSIR (Yang et al., T-IT Sep'15)

Channel Capacity & Optimal Power Allocation

- Capacity:

$$\mathbf{C}(\bar{P}) = \mathbb{E}_{\mathsf{H}}\left[\frac{1}{2}\log\left(1 + \frac{H^2\mathcal{P}_{\mathsf{WF}}(H)}{\sigma_N^2}\right)\right]$$

- Water filling power allocation:

$$\mathcal{P}_{\mathsf{WF}}(H) = \left(\lambda - \frac{\sigma_N^2}{H^2}\right)^+, \ \mathbb{E}_{\mathsf{H}}\Big[\mathcal{P}_{\mathsf{WF}}(H)\Big] = \bar{P}$$

Question: Given a codeword length n, average probability of error ϵ and power constraint $\bar{P}$, what is the back off from $\mathbf{C}(\bar{P})$?

Result

Lower bound on maximal coding rate $R^*(n, \bar{P}, \epsilon)$

- $\mathbf{ST}$ constraint, i.e., $||\mathbf{X}||^2 \leq n\bar{P}$:

$$\boxed{R^*(n, \bar{P}, \epsilon) \geq \mathbf{C}(\bar{P}) + \sqrt{\frac{V_{\mathsf{BF}}(\bar{P})}{n}}\Phi^{-1}(\epsilon) - \frac{c_\epsilon(\bar{P})}{\sqrt{n}} + \frac{\log n}{2n} + O\Big(\frac{1}{n}\Big)}$$

$$V_{\mathsf{BF}}(\bar{P}) \triangleq \mathbb{E}\Big[V(G)\Big] + n_c \mathsf{Var}\Big[C(G)\Big] + \frac{1}{2}\mathsf{Var}\Big[\mathcal{L}(G)\Big]$$

$$\mathcal{L}(x) = \frac{\sigma_N^2}{x+\sigma_N^2}, V(x) = \frac{1-\mathcal{L}^2(x)}{2}, C(x) = \frac{1}{2}\log\left(1 + \frac{x}{\sigma_N^2}\right), G = H^2\mathcal{P}_{\mathsf{WF}}(H)$$

Result

Upper bound on maximal coding rate $R^*(n, \bar{P}, \epsilon)$

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$$V_{\mathsf{BF}}'(\bar{P}) \triangleq \mathbb{E}_{\mathsf{G}}[V(G)] + n_c \mathsf{Var}\left[C(G) + \frac{\bar{P}}{2\lambda} - \frac{\mathcal{L}(G)}{2}\right]$$

Obtaining Lower Bound

1) Decouple Coding & Power Control

- Sample codewords *uniformly* from $\{\mathbf{x}' : ||\mathbf{x}'||^2 = n(1-\delta_n)\}$
- Power control: $X_k = \sqrt{\mathcal{P}_{\mathsf{WF}}(H_k)}X'_k$
- $\mathbf{ST}$ constraint violation at time k: If power transmitted till $k > n\bar{P}$

$$\mathbb{P}\Big[\text{violating } \mathbf{ST} \text{ constraint}\Big] = \mathbb{P}\left[\bigcup_{b=1}^{B}\left\{\sum_{l=1}^{b}||\underline{X}'_l||^2\mathcal{P}_{\mathsf{WF}}(H_l) > Bn_c\bar{P}\right\}\right]$$
$$\leq c_1\exp\Big(-c_2 n\delta_n^2\Big) = \epsilon_n^* < \epsilon.$$

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- Probability of error in decoding: $\epsilon - \epsilon^*$

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2) The $\beta - \beta$ bound

- The $\beta - \beta$ bound [Yang et al., 2016]: For *any* $\tau \in (0, \epsilon')$, $\epsilon' = \epsilon - \epsilon^*$, *any* auxiliary channel $\mathbb{Q}$,

$$M^* \geq \frac{\beta_\tau\left(\tilde{\mathbb{P}}_1, \tilde{\mathbb{P}}_2\right)}{\beta_{1-\epsilon'+\tau}\left(\mathbb{P}_1, \mathbb{P}_2\right)},$$

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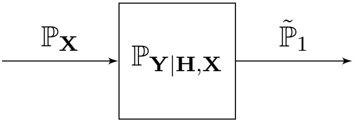

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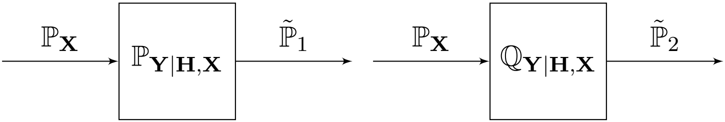

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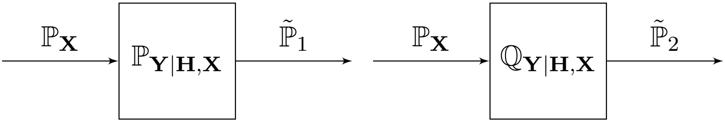

Figure 4: Joint distributions $\mathbb{P}_1$ and $\mathbb{P}_2$ resp.

Obtaining Lower Bound

3) Choosing $\mathbb{Q}$ and bounding β terms

- Fix auxiliary channel $\mathbb{Q}_{\mathbf{Y}\big|\mathbf{X},\mathbf{H}} \equiv \prod_{b=1}^{B}\prod_{k=1}^{n_c}\mathcal{N}\left(0,\sigma_N^2+H_b^2\mathcal{P}_{\mathsf{WF}}(H_b)\right)$
- Lower bound on $\log\beta_\tau\left(\tilde{\mathbb{P}}_1,\tilde{\mathbb{P}}_2\right)$:

$$\log\beta_\tau\left(\tilde{\mathbb{P}}_1,\tilde{\mathbb{P}}_2\right) \geq \tau^{-1}\left[-1-D\left(\tilde{\mathbb{P}}_1||\tilde{\mathbb{P}}_2\right)\right]=O(1).$$

- Upper bound on $\beta_{1-\epsilon'+\tau}\left(\mathbb{P}_1,\mathbb{P}_2\right)$: For appropriate choice of γ_0,

$$\log\beta_{1-\epsilon'+\tau}\left(\mathbb{P}_1,\mathbb{P}_2\right) \leq \log\mathbb{P}_2\left[\frac{d\mathbb{P}_1}{d\mathbb{P}_2}\geq\gamma_0\right]\leq-\frac{\log n}{2}-\log\gamma_0+\text{constant}.$$

Obtaining Upper Bound

1) Meta Converse & Choice of Auxiliary Channel

- Assume non causal CSIT.
- $\mathcal{C}\Big(M, n, \epsilon, \bar{P}\Big)$: code satisfy $\mathbf{ST}$ constraint with equality.
- Fix auxiliary channel $\mathbb{Q}_{\mathbf{Y}\big|\mathbf{X},\mathbf{H}} \equiv \prod_{b=1}^{B}\prod_{k=1}^{n_c}\mathcal{N}\big(0, \sigma_N^2 + H_b^2\mathcal{P}_{\mathsf{WF}}(H_b)\big)$.
- Avg. prob. of error of $\mathcal{C}\Big(M, n, \epsilon, \bar{P}\Big)$ over $\mathbb{Q}$: $\frac{M-1}{M}$
- Meta-converse and its relaxation: For any $\mathbb{Q}$, $\gamma > 0$,

$$M \leq \frac{1}{\beta_{1-\epsilon}\big(\mathbb{P}_{\mathbf{X},\mathbf{H},\mathbf{Y}}, \mathbb{Q}_{\mathbf{X},\mathbf{H},\mathbf{Y}}\big)} \leq \frac{\gamma}{\mathbb{P}\big[\mathcal{I}_\gamma\big] - \epsilon},$$

$$\mathbb{P}\big[I_\gamma\big] = \mathbb{P}\left[\frac{d\mathbb{P}_{\mathbf{X},\mathbf{H},\mathbf{Y}}}{d\mathbb{Q}_{\mathbf{X},\mathbf{H},\mathbf{Y}}} \leq \gamma\right]$$

.

Obtaining Upper Bound

2) Lower bound on $\mathbb{P}\left[\mathcal{I}_\gamma\right]$: Two step approximation

- View as a parallel channel with N sub channels
  - Constraint set $\subset \mathbb{R}^N$ (instead of $\mathbb{R}^n$)
- Using Berry Esseen, for some $\mu,\ \nu$, $c_3 > 0$

$$\mathbb{P}\left[\mathcal{I}_\gamma\right] \geq \mathbb{E}_{\mathbf{H}}\left[\Phi\left(\frac{\log\gamma - \mu(\mathbf{H})}{\sqrt{\nu(\mathbf{H})}}\right)\right] - \frac{c_3}{\sqrt{n}}$$

- $\mathbb{E}_{\mathbf{H}}\left[\Phi\left(\frac{\log\gamma - \mu(\mathbf{H})}{\sqrt{\nu(\mathbf{H})}}\right)\right] \geq \Phi\left(\frac{\log\gamma - n\mathbf{C}(\bar{P})}{\sqrt{nV'_{\text{BF}}(\bar{P})}}\right) - \frac{c_4}{\sqrt{n}}.$
  - Infinite Taylor series, concentration bounds, Berry Esseen
- Choosing $\gamma = n\mathbf{C}(\bar{P}) + \sqrt{nV'_{\text{BF}}(\bar{P})}\Phi^{-1}(\epsilon + \frac{2c_4}{\sqrt{n}})$, the result follows.

Numerical example

Finite blocklength approximation: Rate versus power

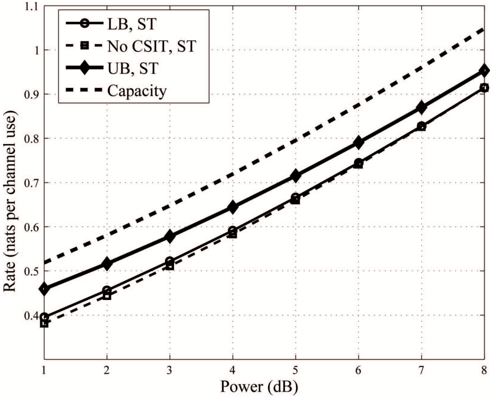

Figure 5: Rate versus power

- Fading distribution: A quantized version of Rayleigh distribution with parameter 0.75, 10 mass points

Concluding remarks

- Result holds for for general fading state space, long term power constraint
- Quest for matching second order coefficients...