$Ax=b$: A Familiar Setup, Axioms and An Open Question

Karthik P. N.

Joint work with Prof. Rajesh Sundaresan

An example

An example

- Consider a problem of image reconstruction

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- Image is represented as a function

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$$f = \sum_{j=1}^{n} x_j f_j, \quad f_j = \text{ indicator of the } j\text{th pixel}$$

- Measurements

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$$✓A \qquad x \qquad b✓$$

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$$A \qquad x \qquad b$$

- Goal: Recover $x = (x_1, \ldots, x_n)^T$ from the measurements

Criterion for / goodness of recovery

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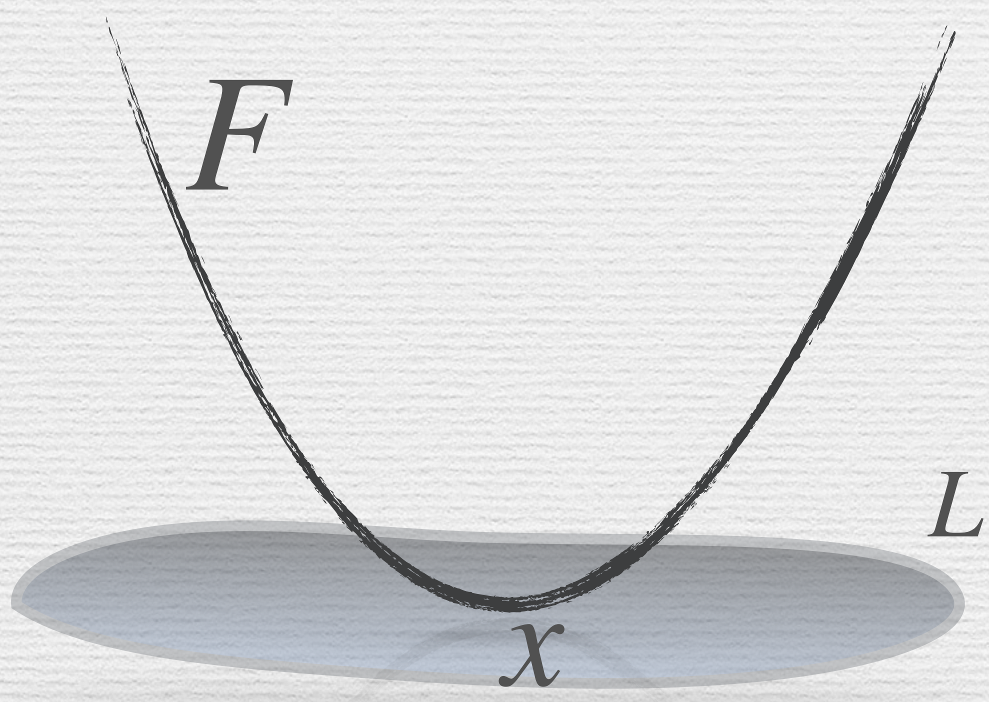

$$L = \{w \in \mathbb{R}^n : Aw = b\}$$

$$x = \arg\min_{w \in L} F(w)$$

Function-minimisation approach: examples

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$$x = \arg\min_{w \in L \cap \mathbb{R}^n_+} \sum_{i=1}^{n} w_i \log w_i$$

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- I-divergence minimisation with respect to a nonnegative prior guess

$$x = \arg \min_{w \in L \cap \mathbb{R}_+^n} \sum_{i=1}^{n} w_i \log \frac{w_i}{x_i^\star} - w_i + x_i^\star$$

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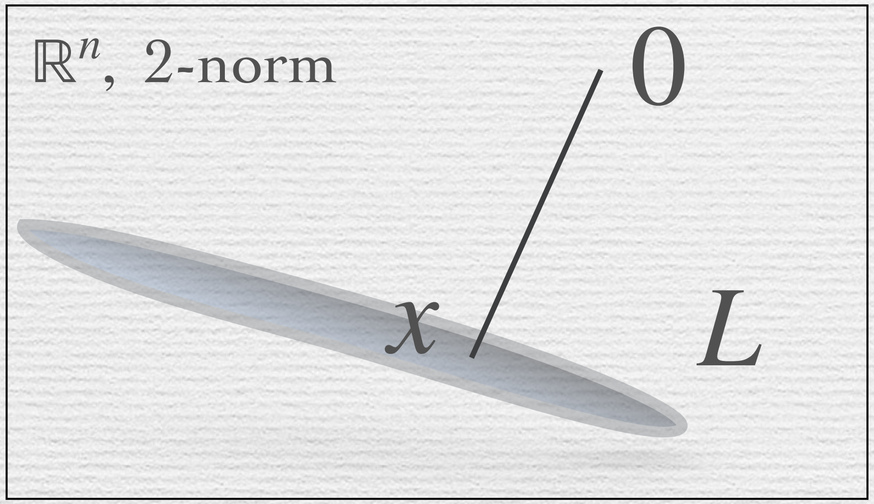

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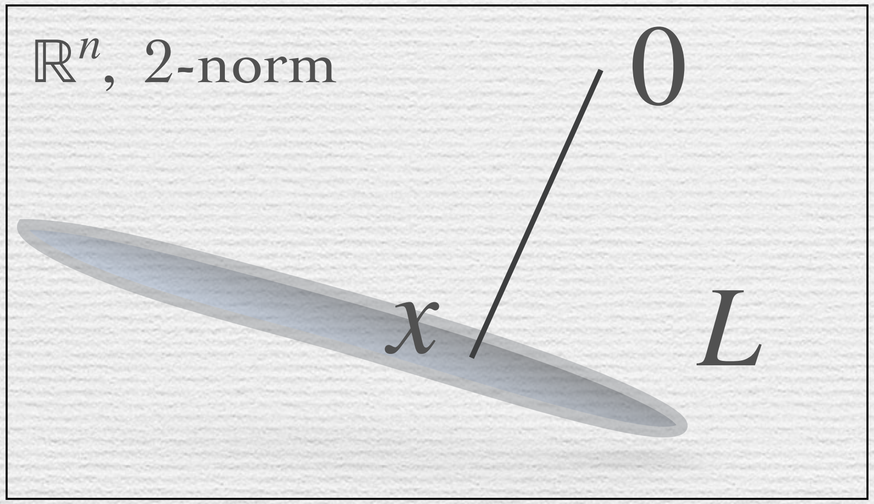

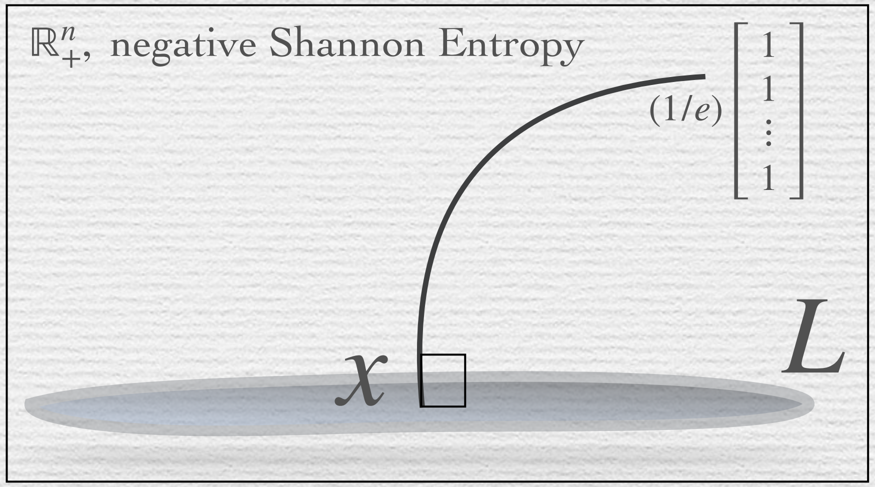

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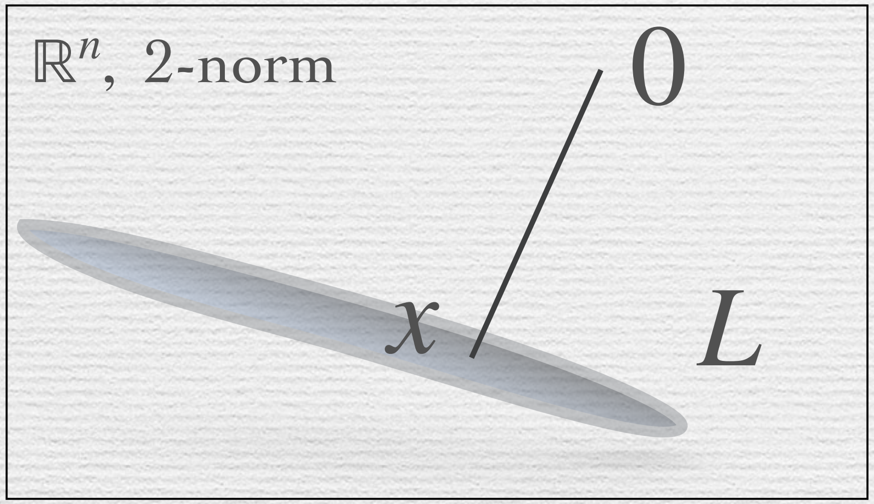

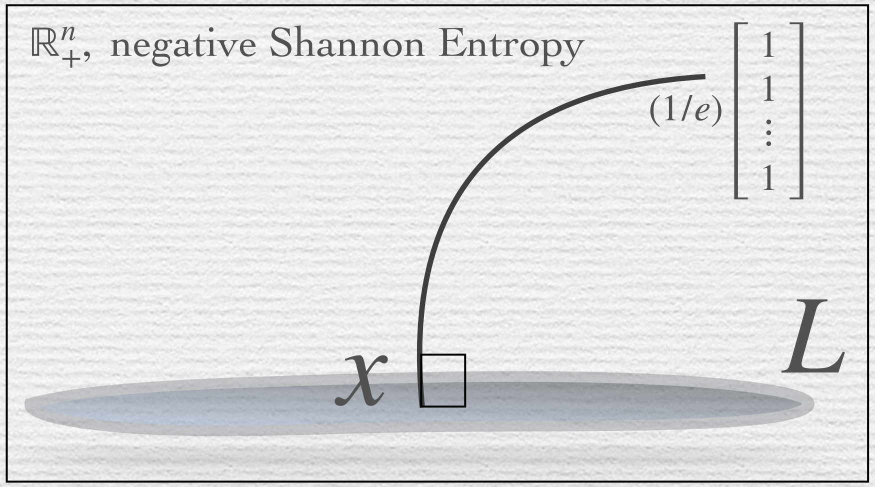

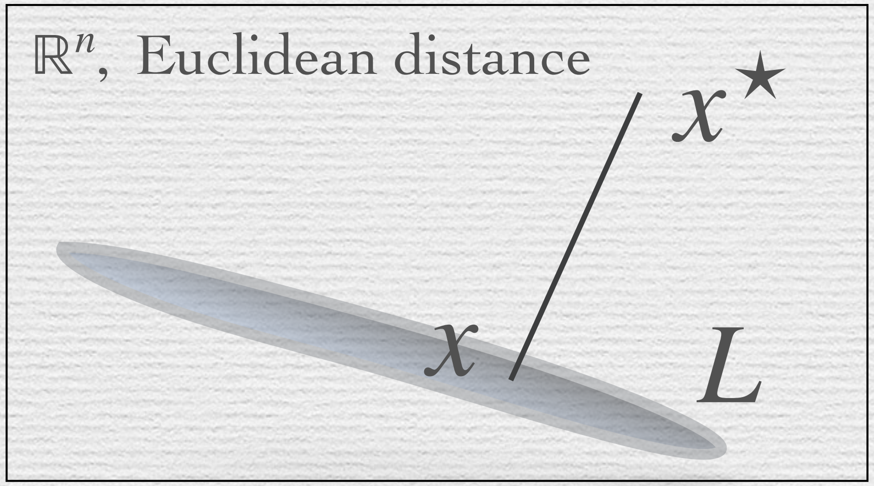

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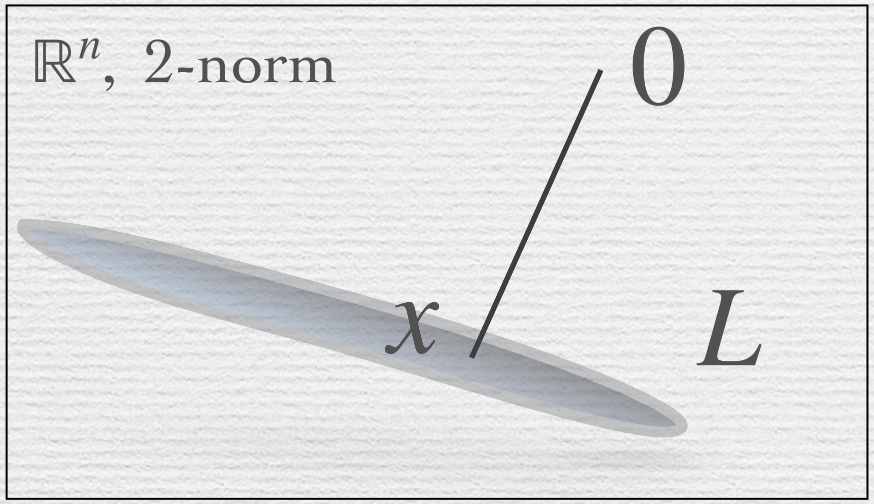

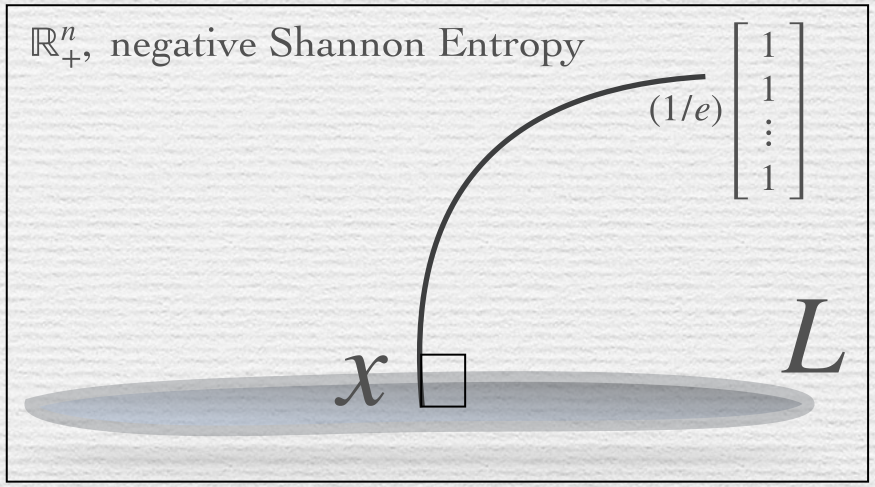

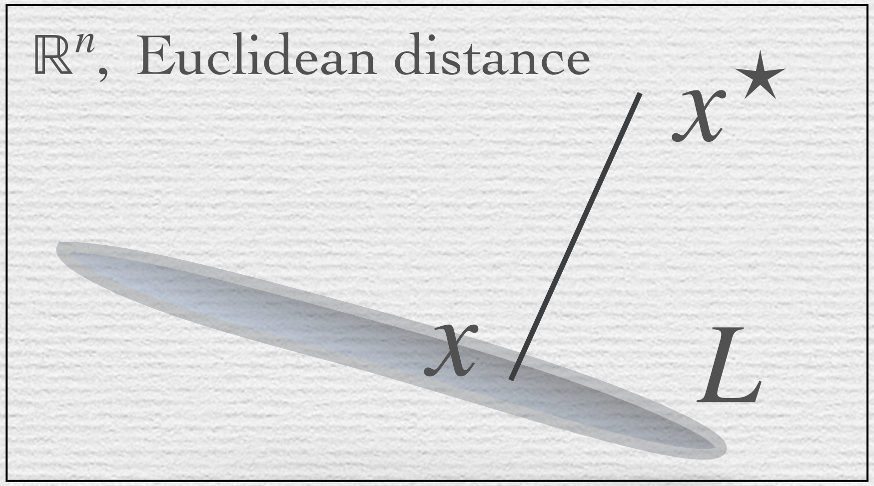

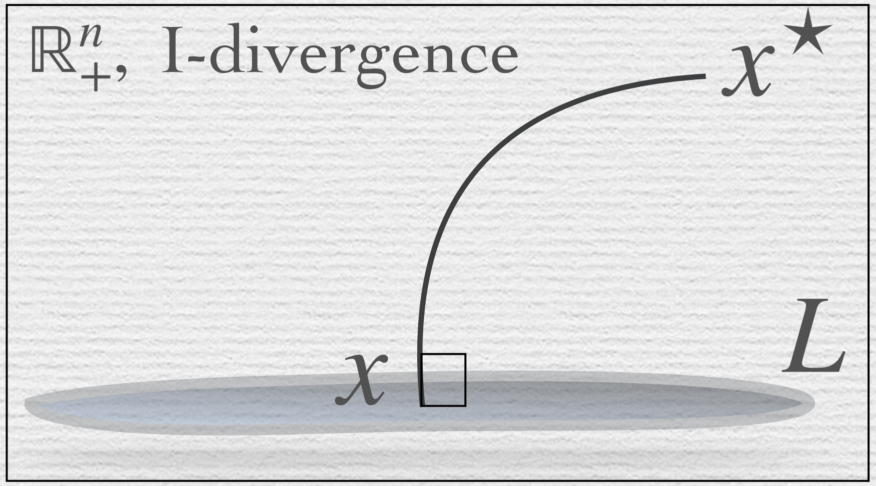

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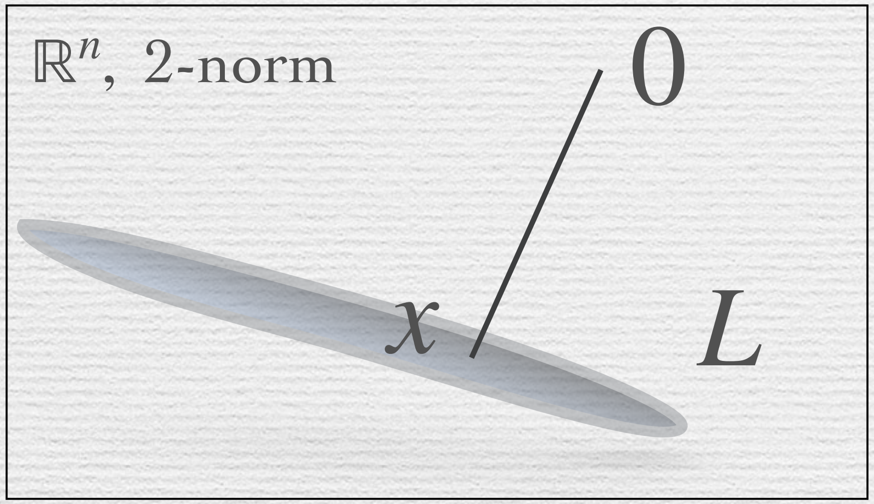
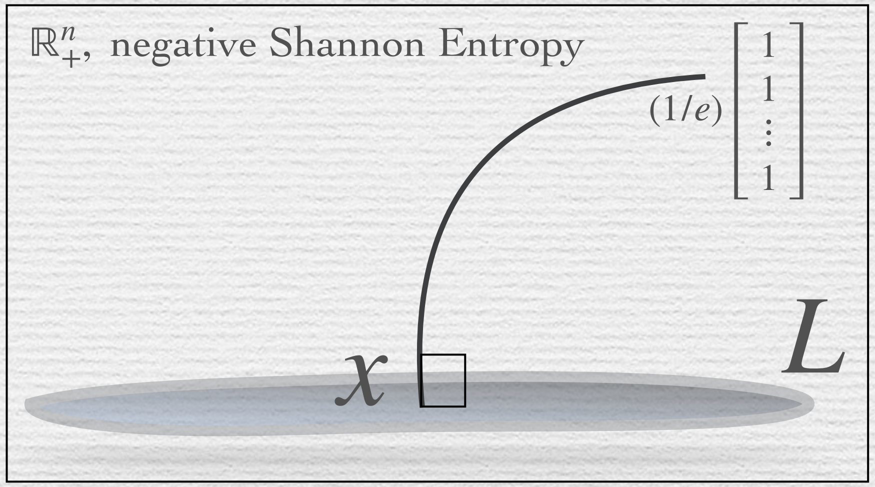
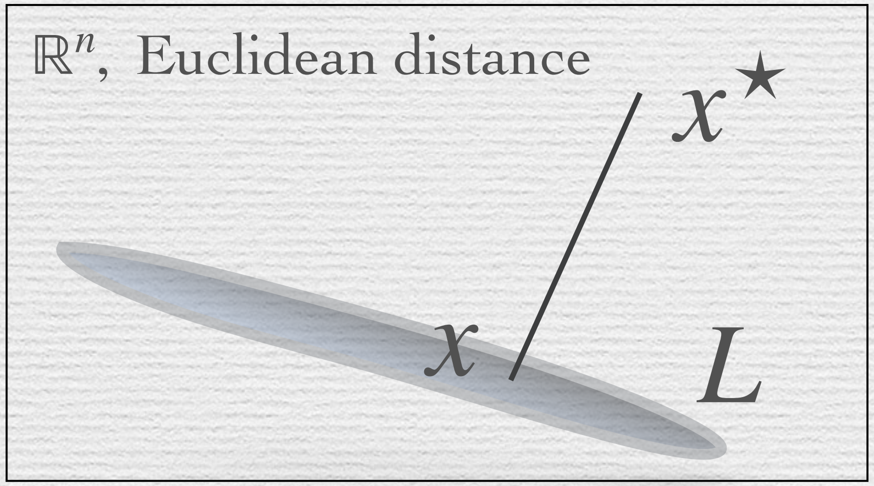
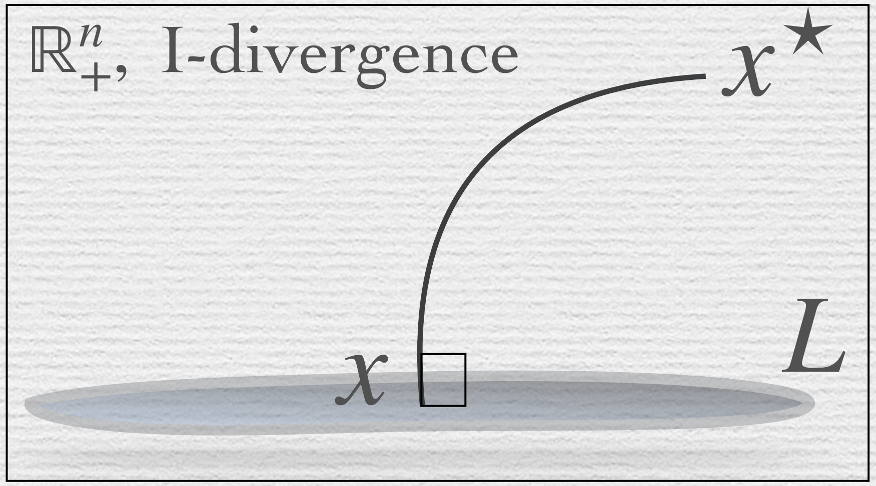




Thus, function-minimisation defines a projection rule





Projection rules: formal definition

- Let

$$\mathscr{L} = \left\{ L = \{w \in \mathbb{R}^n : Aw = b\} : A \text{ is a } k \times n \text{ matrix having rank } k,\ b \in \mathbb{R}^k \right\}$$

where $k = 0,1,2,\ldots,n$

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$$(L, x^\star) \mapsto \Pi(L \mid x^\star) \in L$$

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$$\text{If } x^\star \in L, \text{ then } \Pi(L \mid x^\star) = x^\star$$

Projection Rules and Function Minimisation

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$$\begin{aligned} F : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x^\star, w) &\mapsto F(w \,|\, x^\star) \end{aligned}$$

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Definition: Π is generated by F, if

$$\Pi(L \,|\, x^\star) = \arg\min_{w \in L} F(w \,|\, x^\star)$$

Is every projection rule generated by some function?

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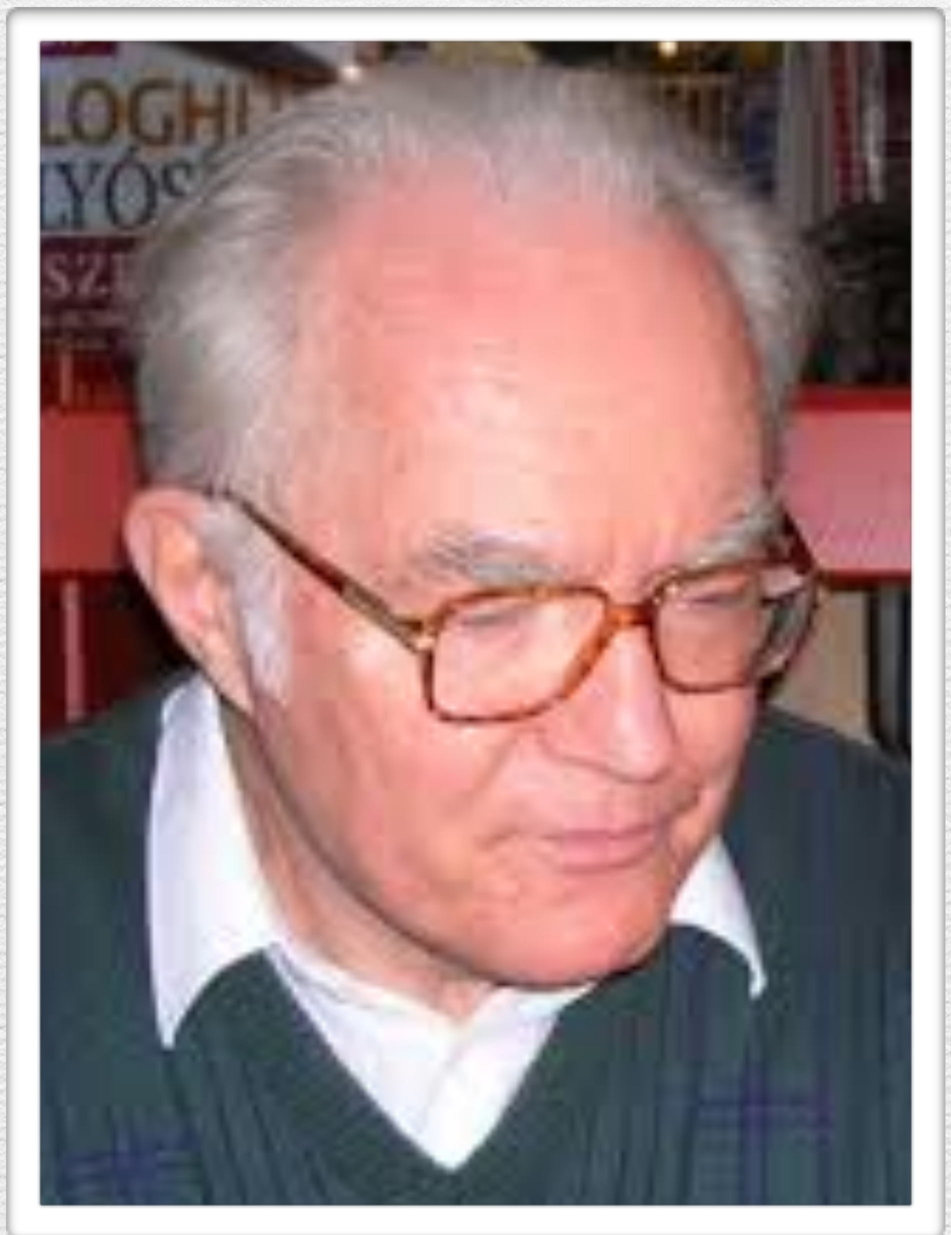

The Annals of Statistics
1991, Vol. 19, No. 4, 2032–2066

WHY LEAST SQUARES AND MAXIMUM ENTROPY? AN AXIOMATIC APPROACH TO INFERENCE FOR LINEAR INVERSE PROBLEMS[1]

BY IMRE CSISZÁR

Mathematical Institute of the Hungarian Academy of Sciences

An attempt is made to determine the logically consistent rules for selecting a vector from any feasible set defined by linear constraints, when either all n-vectors or those with positive components or the probability vectors are permissible. Some basic postulates are satisfied if and only if the selection rule is to minimize a certain function which, if a "prior guess" is available, is a measure of distance from the prior guess. Two further natural postulates restrict the permissible distances to the author's f-divergences and Bregman's divergences, respectively. As corollaries, axiomatic characterizations of the methods of least squares and minimum discrimination information are arrived at. Alternatively, the latter are also characterized by a postulate of composition consistency. As a special case, a derivation of the method of maximum entropy from a small set of natural axioms is obtained.

Axiomatic approach

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- Csiszár demonstrated that if projection rules satisfy some naturally appealing axioms, they must be generated by "nice" functions

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Axioms satisfied by a projection rule	Nature of the function generating the projection rule
Regularity + Locality	$F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star)$, f_i continuously differentiable and strictly convex
Regularity+ Locality + Subspace Transitivity	$F(w \mid x^\star) =$ Bregman's divergence
Regularity+ Locality + Subspace Transitivity + Statistical	$F(w \mid x^\star) =$ I-divergence
Regularity + Locality + Subspace Transitivity + Location Invariance + Scale Invariance	$F(w \mid x^\star) =$ Euclidean distance

The axiom of regularity

$$\mathscr{M} = \left\{ \{w \in \mathbb{R}^n : a^T w = b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R} \right\}$$

The axiom of regularity

$$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n \text{ projection rule}$$

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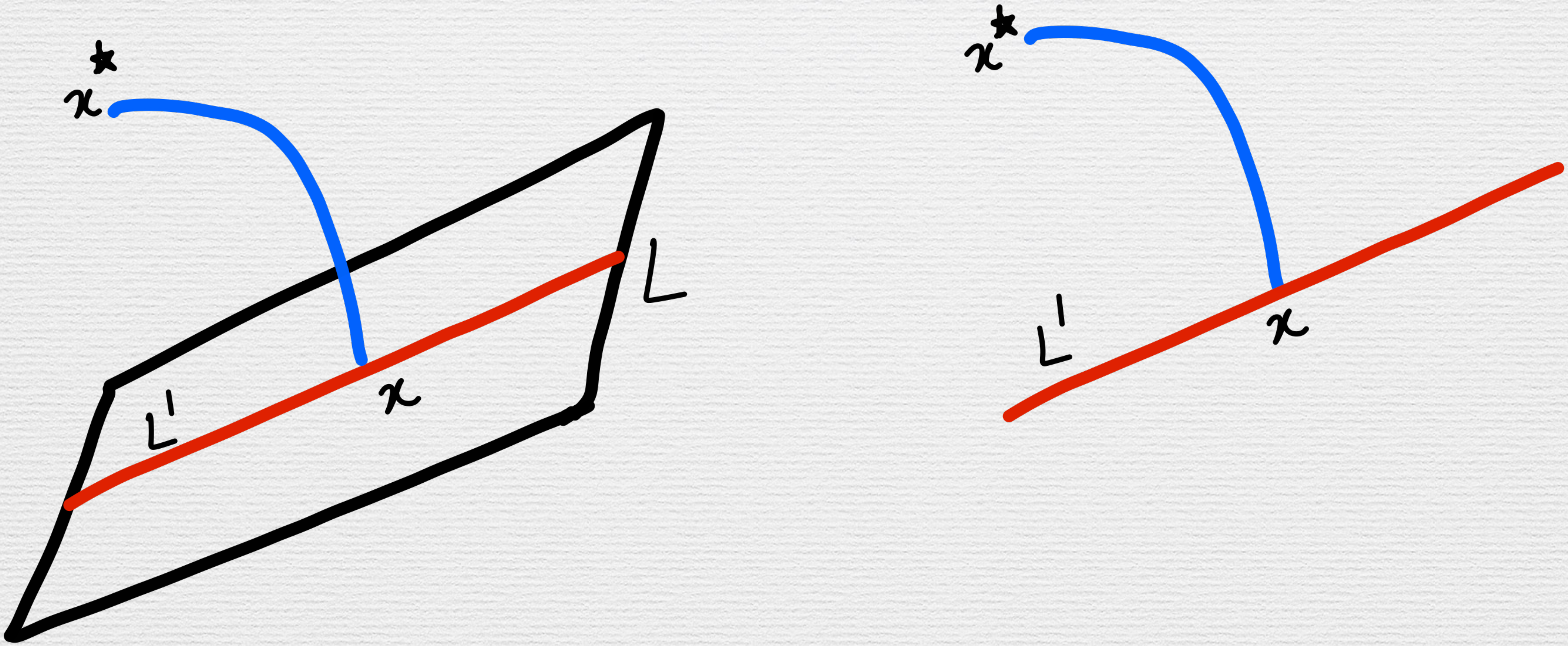
$\Pi : \mathscr{L} \times \mathbb{R}^n \to \mathbb{R}^n$ projection rule

$$\mathscr{M} = \left\{ \{w \in \mathbb{R}^n : a^T w = b\},\ a \in \mathbb{R}^n,\ a \neq 0,\ b \in \mathbb{R} \right\}$$

Π satisfies regularity if, for all $x^\star \in \mathbb{R}^n$,

Regularity	(Consistency) $L' \subset L,\ \Pi(L \mid x^\star) \in L' \implies \Pi(L' \mid x^\star) = \Pi(L \mid x^\star)$
	(Distinctness) $L, \tilde{L} \in \mathscr{M},\ L \neq \tilde{L},\ x^\star \notin L \cap \tilde{L} \implies \Pi(L \mid x^\star) \neq \Pi(\tilde{L} \mid x^\star)$
	(Continuity) $\Pi(\,\cdot \mid x^\star)$ restricted to any fixed dimension is continuous

The axiom of consistency



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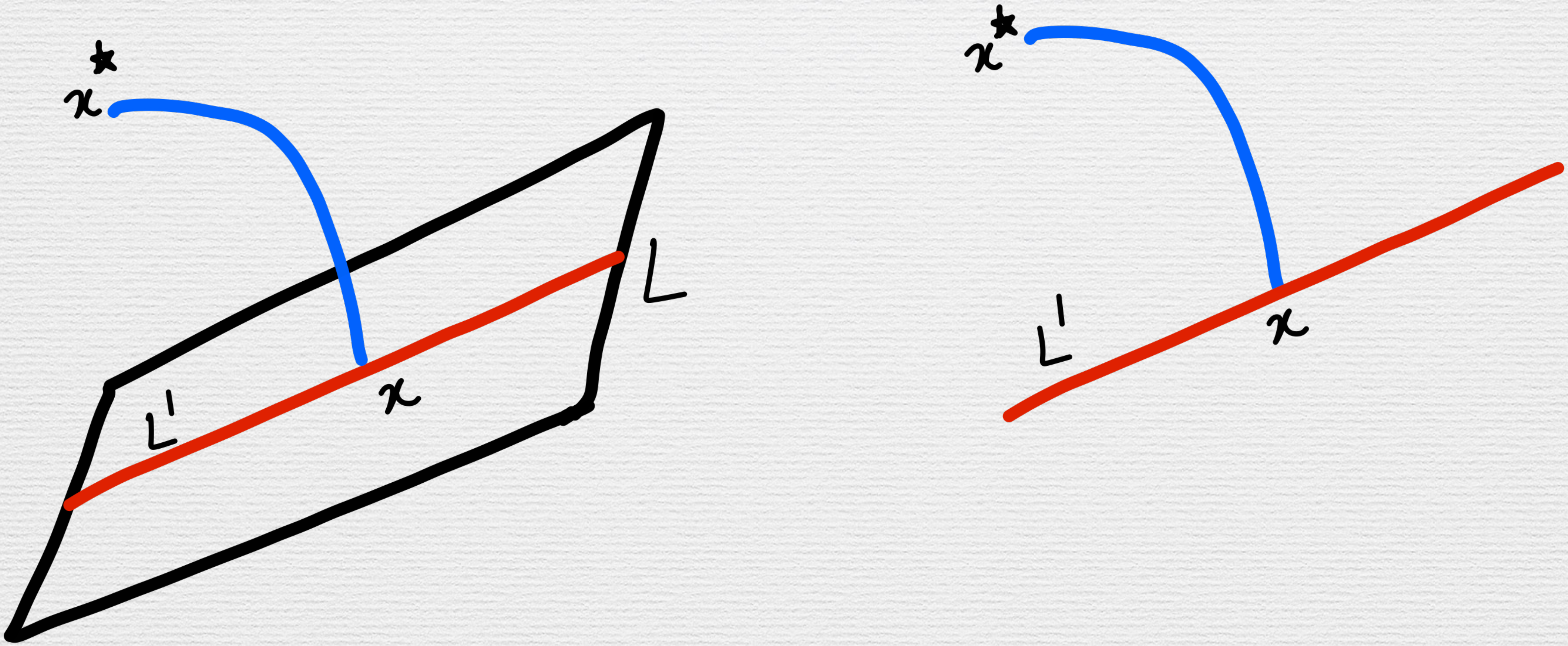

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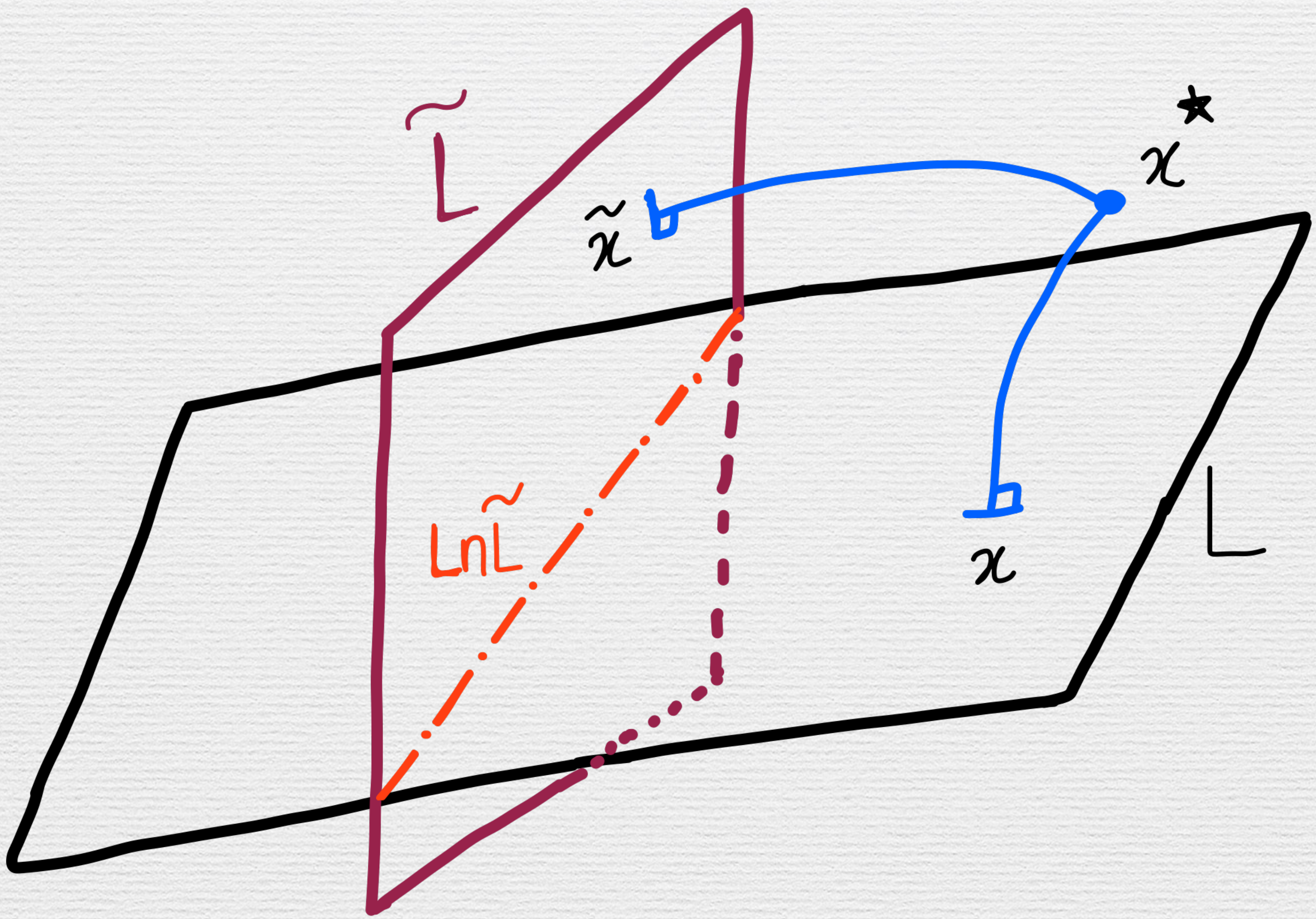
Additional observations do not give a reason to change the original selection

The axiom of distinctness



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- Topology on Y is the quotient topology derived out of the Euclidean topology on X

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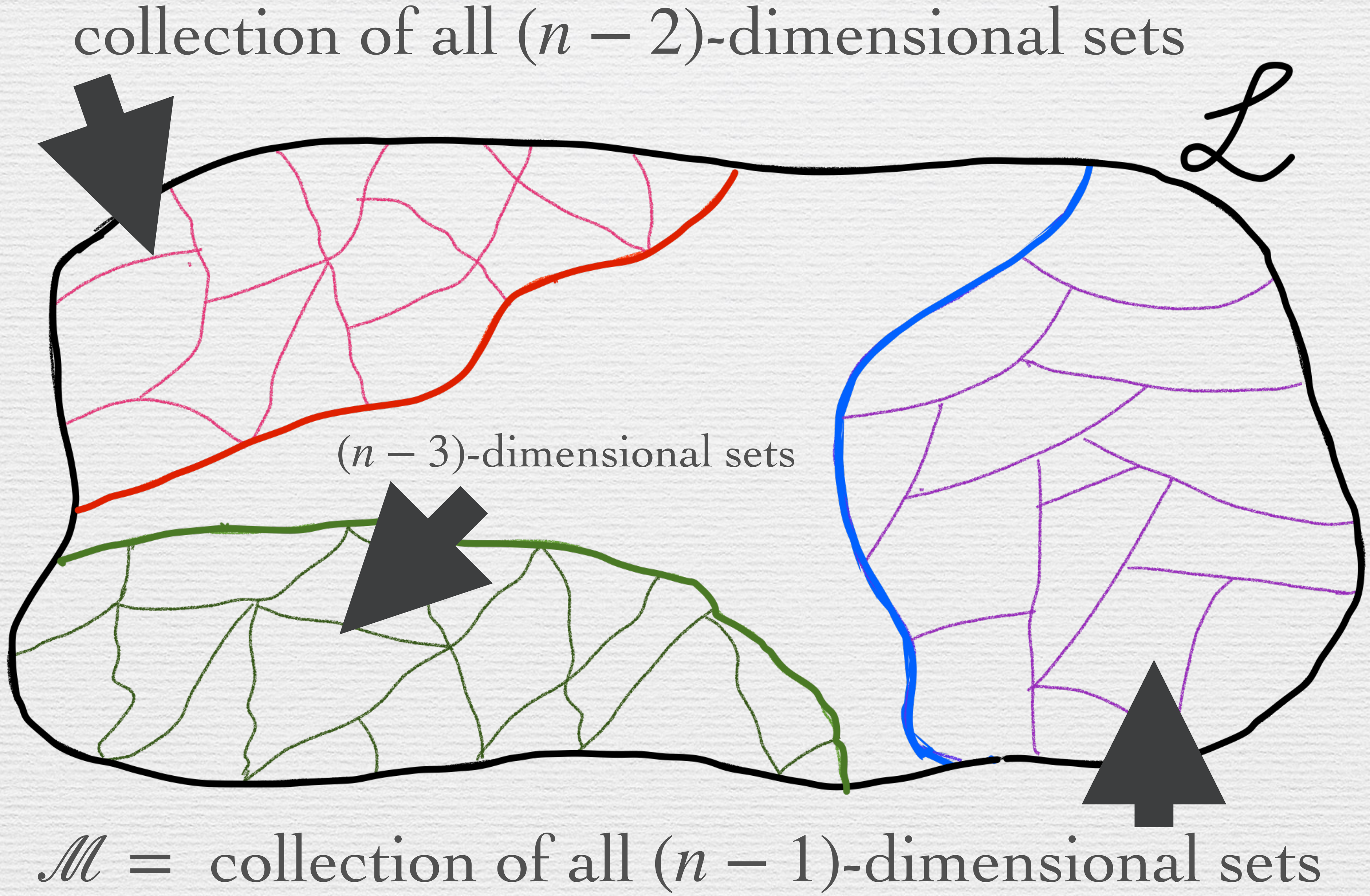
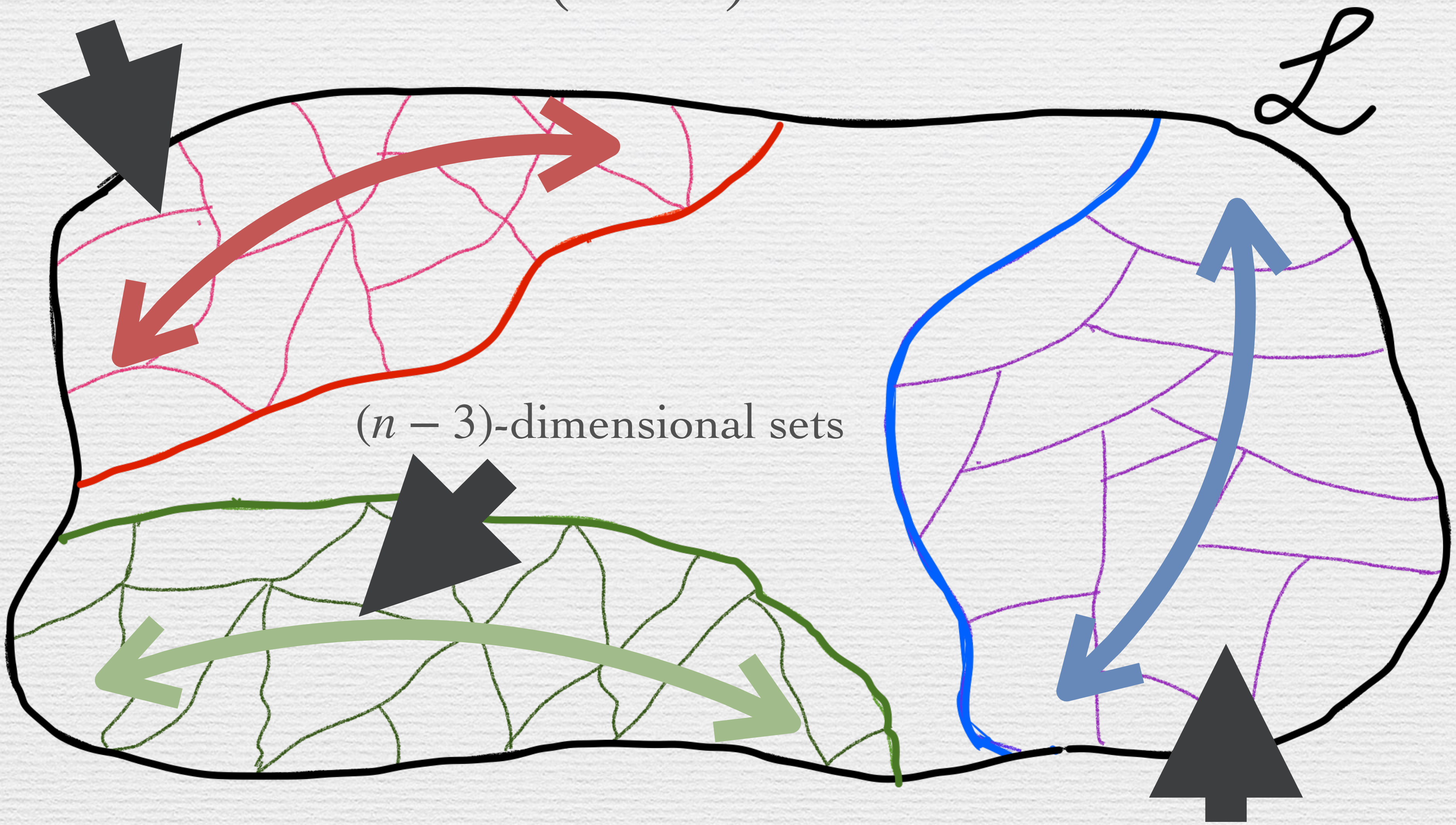


The axiom of continuity

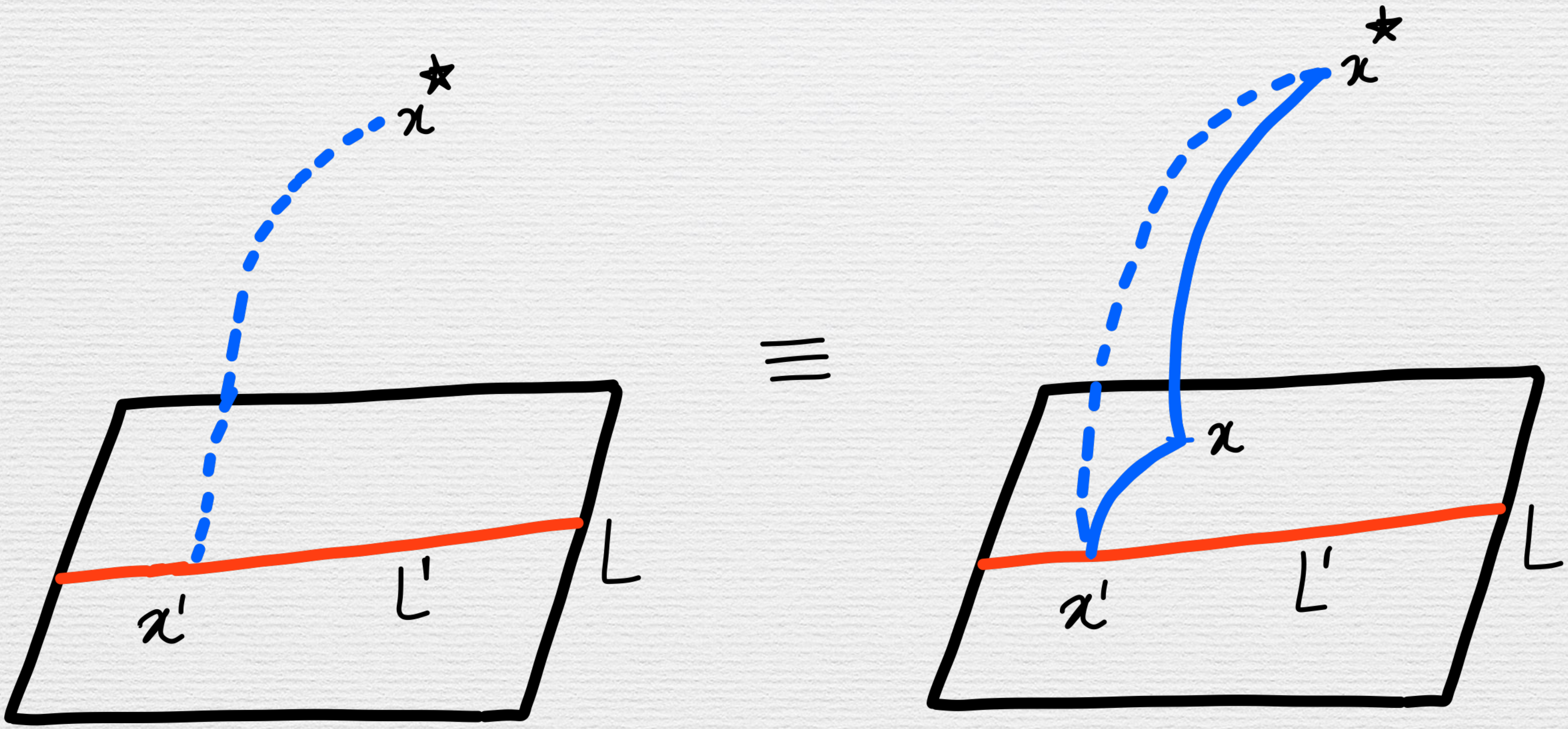
collection of all $(n-2)$-dimensional sets



$\mathscr{M} =$ collection of all $(n-1)$-dimensional sets

$\Pi(\cdot \mid x^\star)$ restricted to any fixed dimension is continuous

Axiom of subspace transitivity



$$L' \subset L \implies \Pi(L' \,|\, x^\star) = \Pi(\Pi(L \,|\, x^\star) \,|\, x^\star)$$

Why are these axioms of interest to us?

Kumar, M. Ashok, and Rajesh Sundaresan. "Minimization Problems Based on Relative $\alpha $-Entropy I: Forward Projection." *IEEE Transactions on Information Theory* 61.9 (2015): 5063-5080.

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Locality	✗

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Locality	✗

- These projection rules are generated by a divergence known as Sundaresan's divergence (relative α-entropy)

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Sundaresan’s divergence

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$$\mathscr{I}_\alpha(w, x) = \frac{\alpha}{1-\alpha} \log \left(\sum_{i=1}^{n} \frac{w_i}{||w||} \left(\frac{x_i}{||x||} \right)^{\alpha - 1} \right),$$

$$\text{where } ||x|| = \left(\sum_{i=1}^{n} x_i^\alpha \right)^{1/\alpha}, \quad \alpha > 0, \alpha \neq 1$$

Sundaresan's divergence

- A parametric generalisation of KL divergence (or I-divergence)

$$\mathcal{I}_\alpha(w, x) = \frac{\alpha}{1-\alpha} \log \left(\sum_{i=1}^{n} \frac{w_i}{||w||} \left(\frac{x_i}{||x||} \right)^{\alpha-1} \right),$$

$$\text{where } ||x|| = \left(\sum_{i=1}^{n} x_i^\alpha \right)^{1/\alpha}, \quad \alpha > 0, \alpha \neq 1$$

- This is **not a Bregman's divergence**. Hence, the corresponding projection rule is not local

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Remainder of the talk is about regularity, and more….

What we want to know about regularity

- Csiszár's results provide a necessary and sufficient axiomatic characterisation of many projection rules

Regularity + Locality ⟷ Projection rule generated by $F(w \mid x^\star) = \sum_{i=1}^{n} f_i(w_i \mid x_i^\star)$

Regularity + Locality + Subspace Transitivity ⟷ Projection rule generated by Bregman's divergence

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Regularity ⟷ Projection rule generated by ???

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- Recall

$$\mathscr{M} = \left\{ \{w \in \mathbb{R}^n : a^T w = b\},\, a \in \mathbb{R}^n,\, a \neq 0,\, b \in \mathbb{R} \right\}$$

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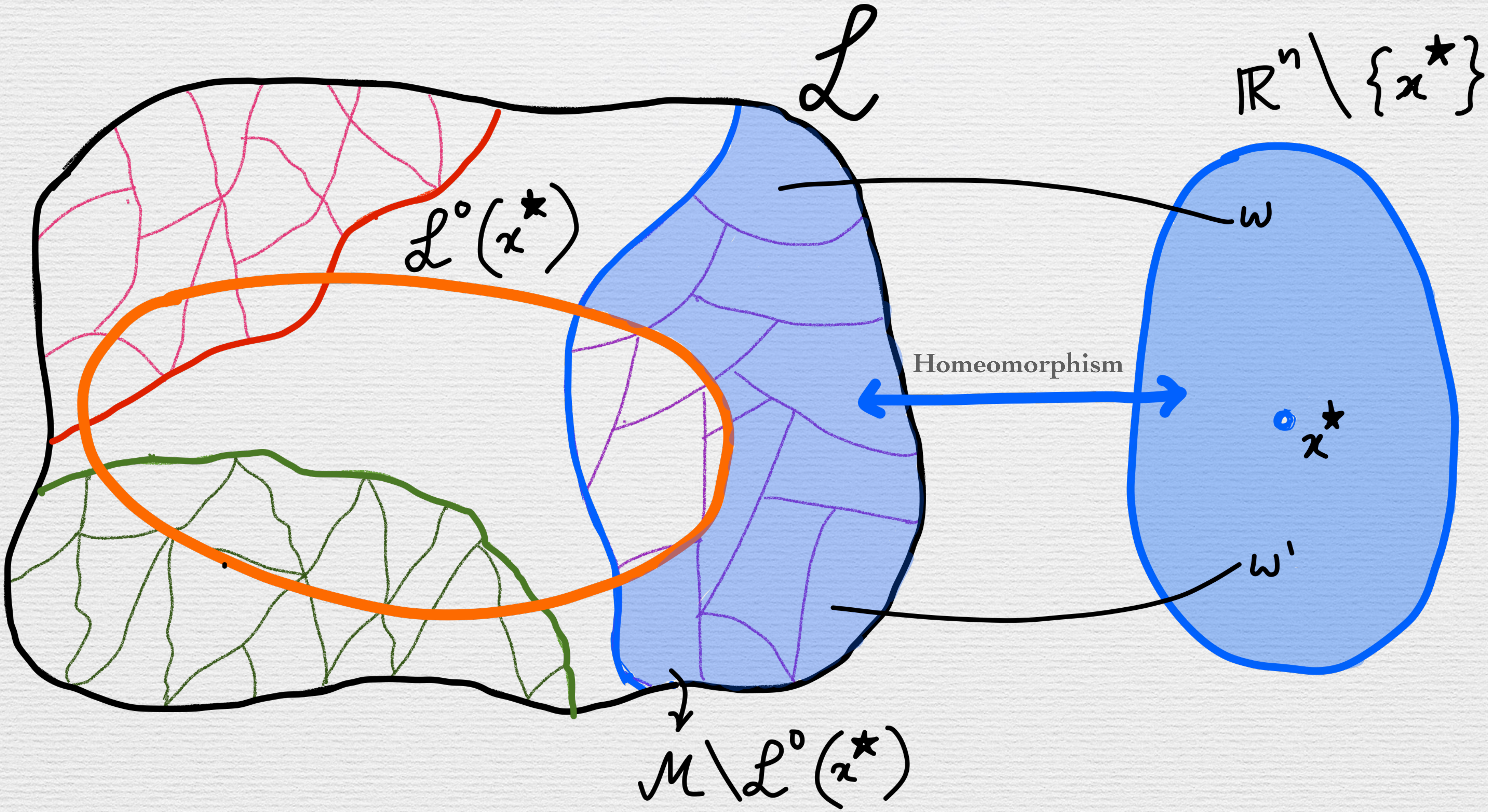

Implications of regularity

This means that for every $w \neq x^\star$,

there exists a unique set L of dimension $(n-1)$

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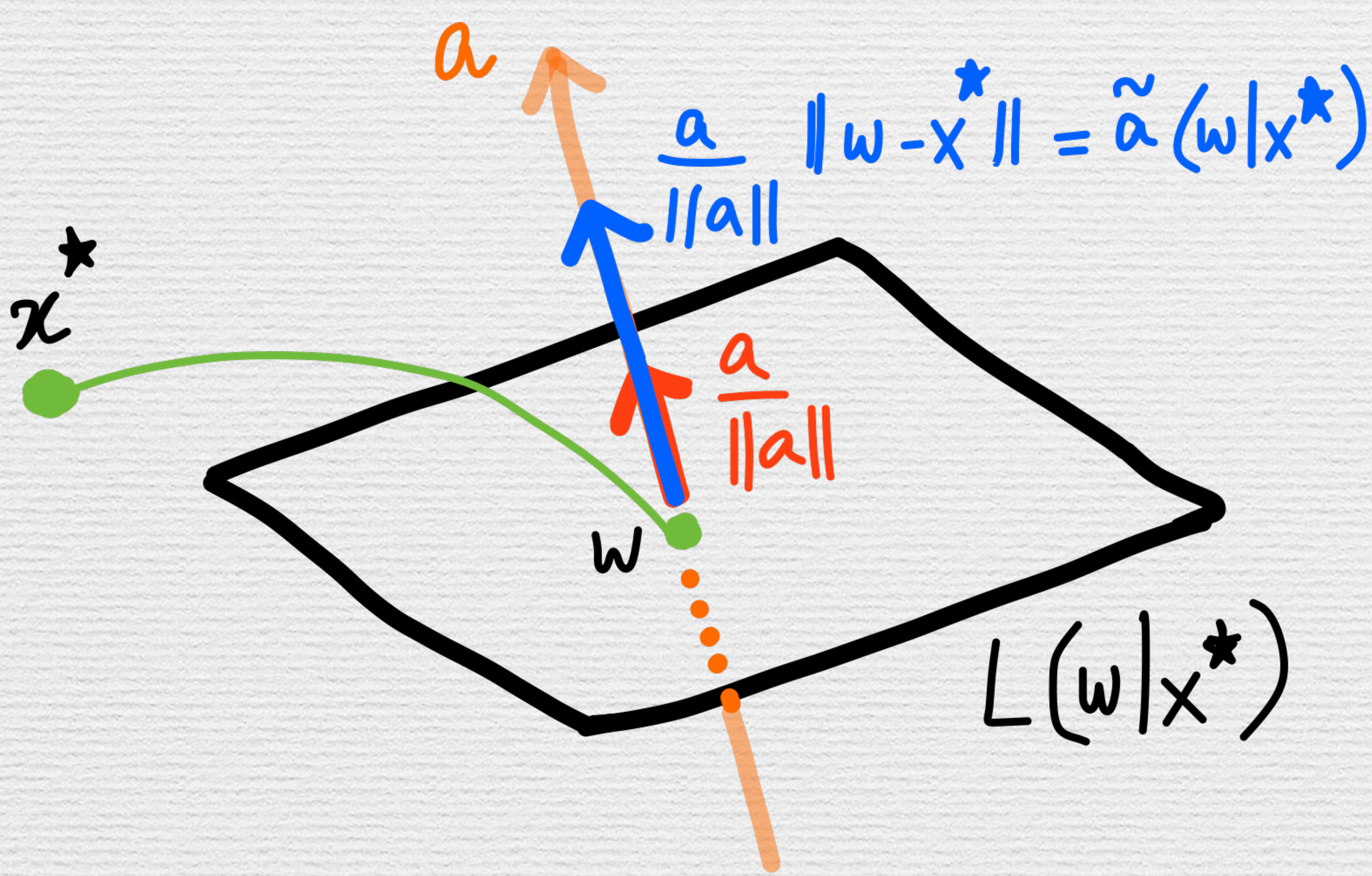

$$
\begin{aligned}
L(w|x^\star) &= \{ y \in \mathbb{R}^n : a^T y = b \} \\
&= \{ y \in \mathbb{R}^n : a^T y = a^T w \} \\
&= \{ y \in \mathbb{R}^n : \tfrac{a}{\|a\|}^T (y - w) = 0 \} \\
&= \{ y \in \mathbb{R}^n : \tilde{a}(w|x^\star)^T (y - w) = 0 \}
\end{aligned}
$$

Regularity and vector fields

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$$\tilde{a}(\,\cdot\,|\,x^\star) : \mathbb{R}^n \to \mathbb{R}^n$$

$$\tilde{a}(\,\cdot\,|\,x^\star) \text{ continuous}$$

$$\tilde{a}(x^\star\,|\,x^\star) = 0,$$

$$\tilde{a}(w\,|\,x^\star) \neq 0 \text{ for all } w \neq x^\star$$

Csiszár's results and vector fields

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Regularity
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This particular form is due to <u>locality</u> axiom

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Does there exist a nonzero scaling $\lambda(\,\cdot\,|x^\star)$
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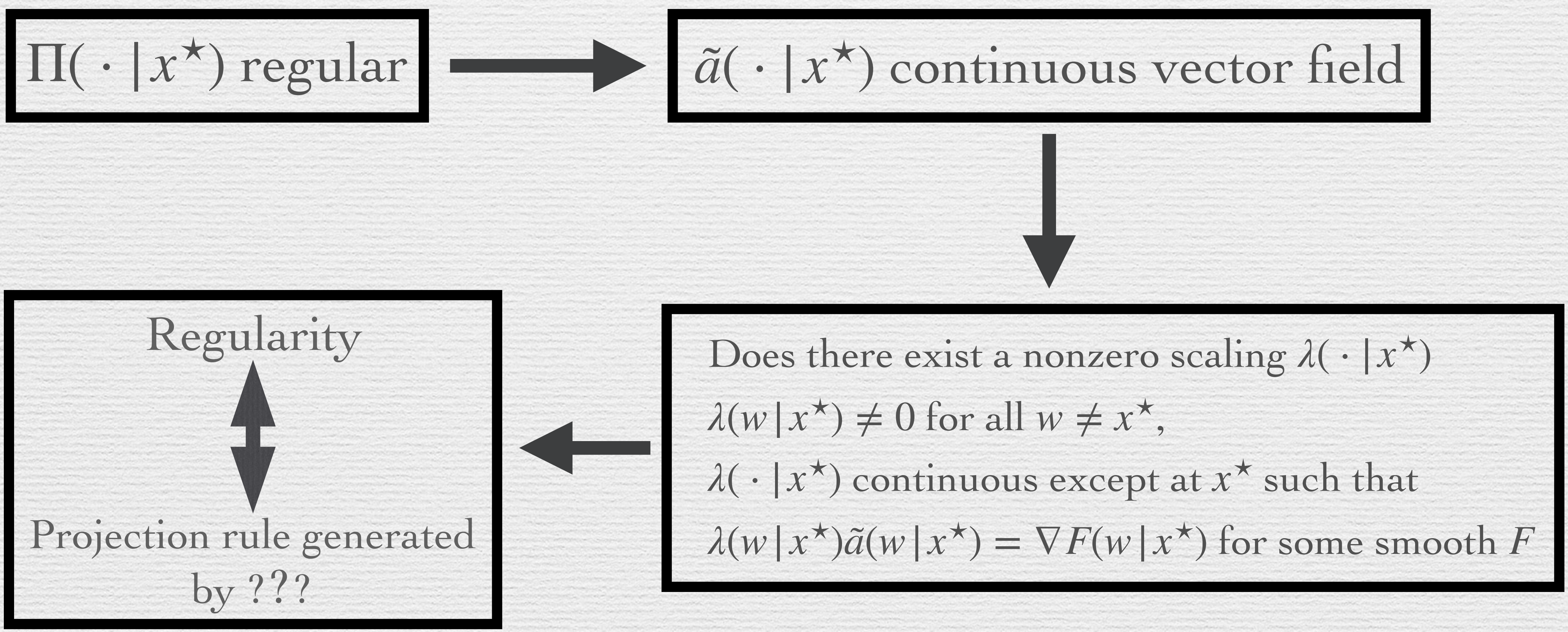

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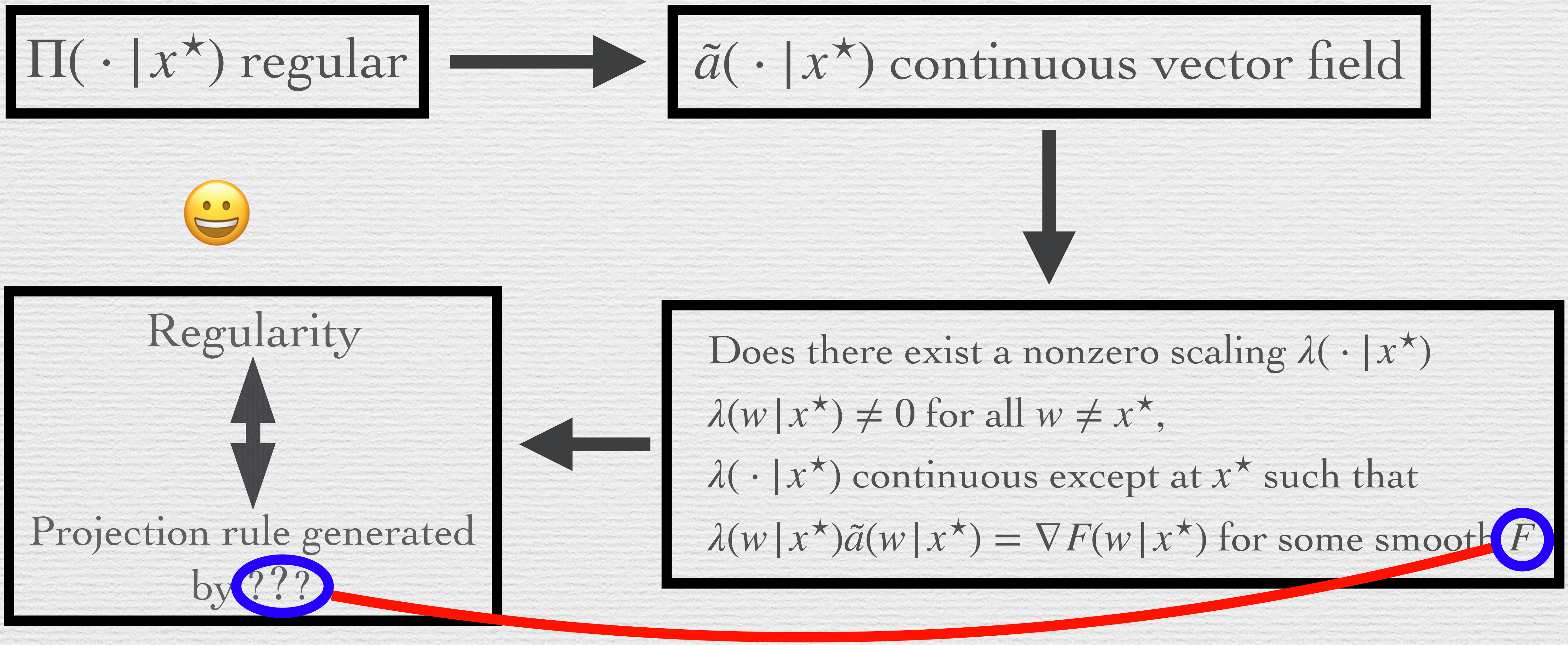

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Open question

Does there exist a nonzero scaling $\lambda(\,\cdot\,|\,x^\star)$

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Regularity

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Projection rule generated by ???

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Given a continuous vector field, does there exist a continuous scaling function such that the scaled vector field is conservative?

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$$a : \mathbb{R}^n \to \mathbb{R}^n \text{ continuous}$$

$$\lambda : \mathbb{R}^n \to \mathbb{R} \text{ continuous}$$

$$\lambda \cdot a = \nabla F?$$

Under more assumption

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$$a_1(w)\frac{\partial \lambda(w)}{\partial w_2} - a_2(w)\frac{\partial \lambda(w)}{\partial w_1} = \lambda(w)\left(\frac{\partial a_2(w)}{\partial w_1} - \frac{\partial a_1(w)}{\partial w_2}\right),$$
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$$a_3(w)\frac{\partial \lambda(w)}{\partial w_1} - a_1(w)\frac{\partial \lambda(w)}{\partial w_3} = \lambda(w)\left(\frac{\partial a_1(w)}{\partial w_3} - \frac{\partial a_3(w)}{\partial w_1}\right)$$

Summary

- Axiomatic characterisation of projection rules
- Regularity: a fundamental axiom
- Regularity has connections with conservative vector fields
- Given a continuous vector field that is not necessarily conservative, is there a continuous scaling that can make the product vector field conservative? Open!

Thank you

The axiom of locality

- Consider two sets L and $\tilde{L}$ of the form

$$L = \{w \in \mathbb{R}^n : w_J \in L_0, w_{J^c} \in L'\}$$

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where $J \subset \{1,2,\ldots,n\}$ is arbitrary

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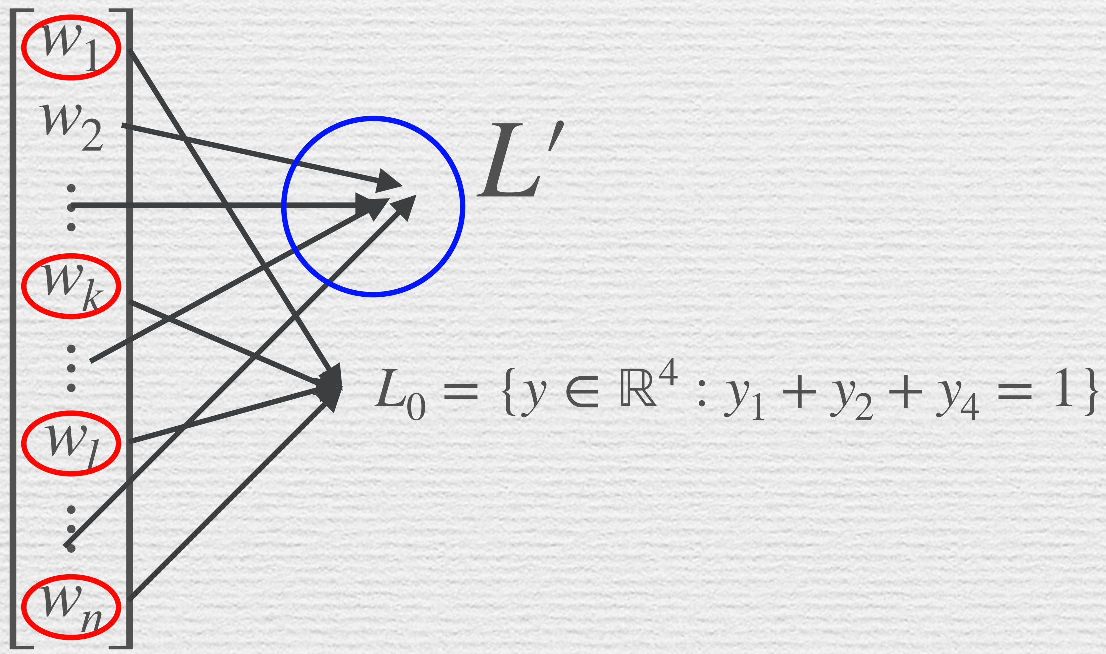

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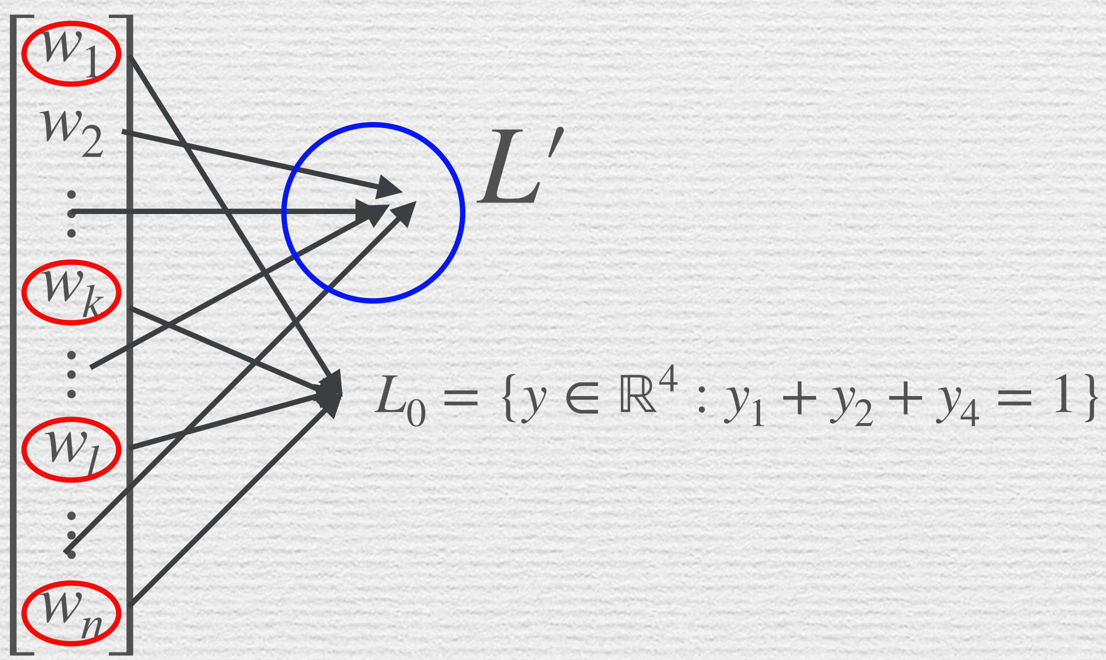

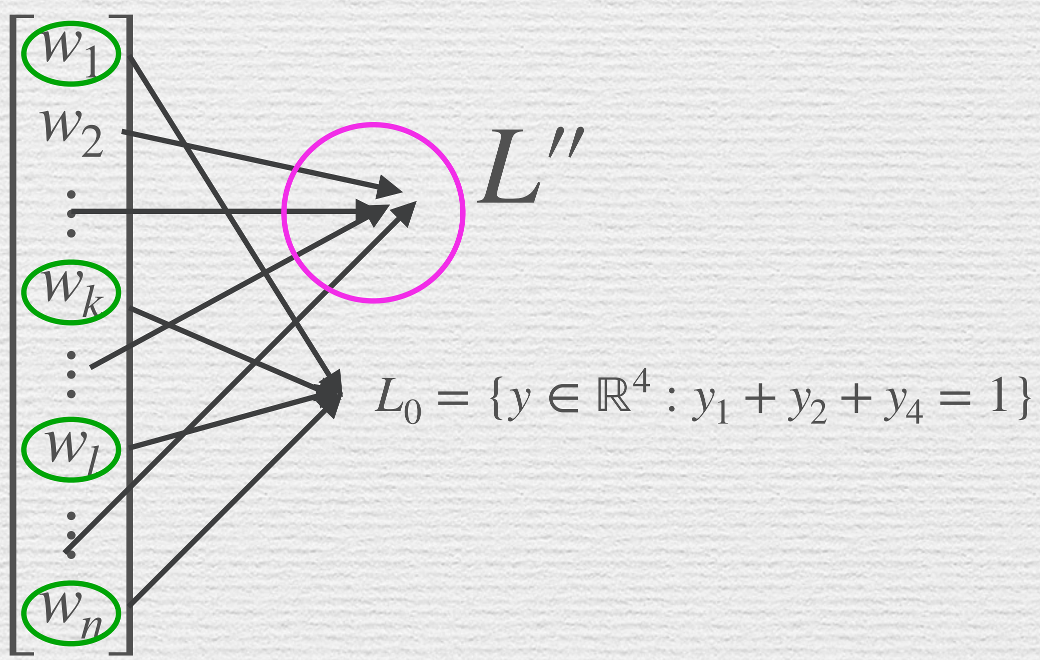

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Then, $(\Pi(L\,|\,x^\star))_J = (\Pi(\tilde{L}\,|\,x^\star))_J$