Rate-Optimal Streaming Codes for Channels with Burst and Isolated Erasures

M. Nikhil Krishnan
(Advisor: P. Vijay Kumar)

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August 23, 2018

Coding for Communication with Delay Constraints
(Codes for Streaming)

Skype

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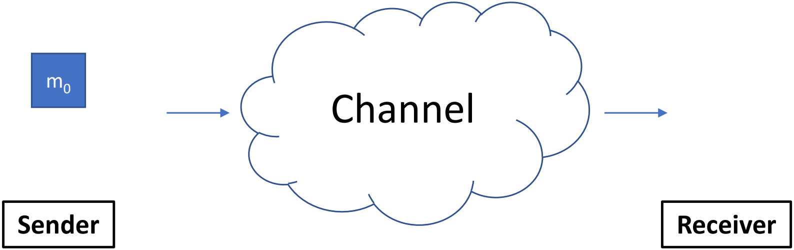
Streaming Applications: Need for Coding



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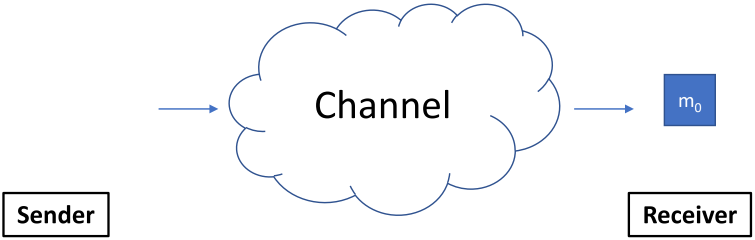

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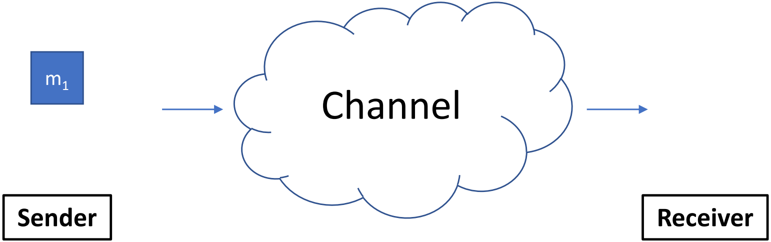

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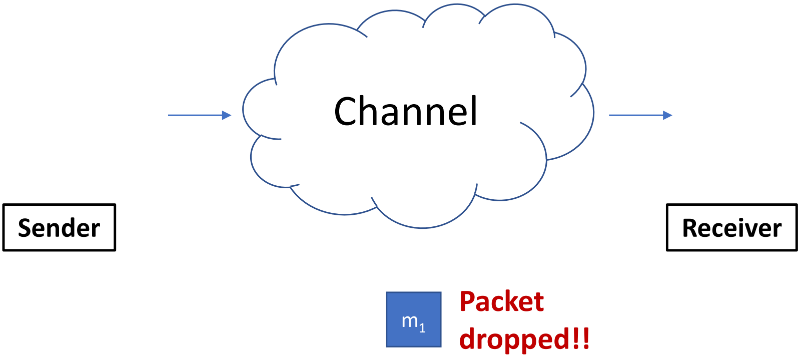

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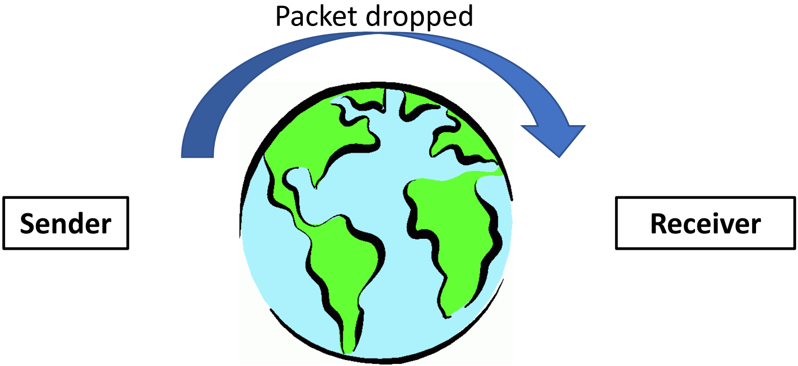

- Delay at least 3x70 milli sec!
- Delay too large for interactive applications
- There is need for coding

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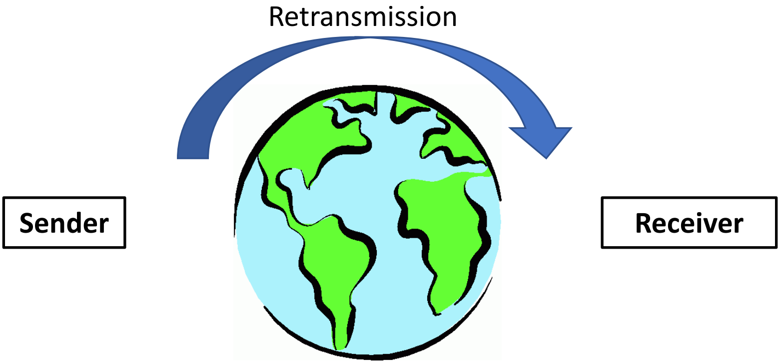
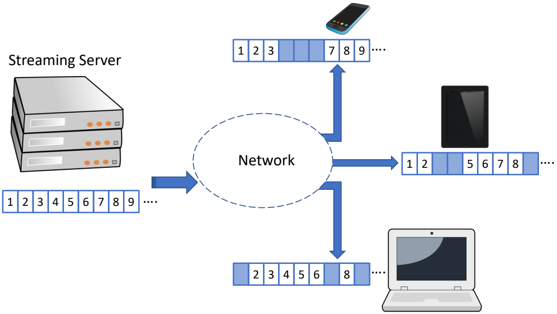


- Delay at least 3x70 milli sec!
- Delay too large[1] for interactive applications
- There is need for coding

[1] International Telecommunication Union recommends the end-to-end latency in interactive voice/video applications to be $<$ 150 milli sec

Codes for Streaming: Overview

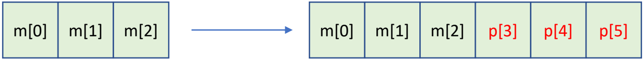
- A continuous stream of message packets $\mathbf{s}[0], \mathbf{s}[1], \ldots$ to be encoded and sent over an erasure channel[2].
- Each coded packet to be decoded with a delay of at most T.



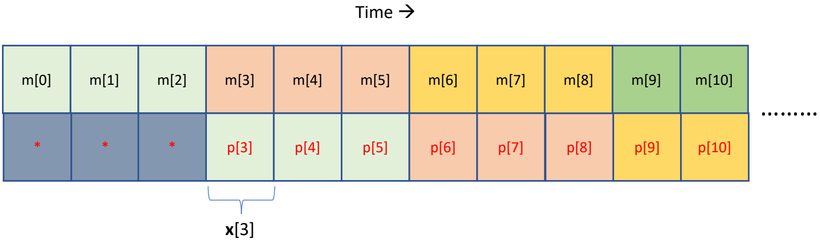
[2] blue rectangles indicate erased packets

Why not traditional codes, say MDS?

- Assume there is a channel which introduces a burst erasure of length ≤ 3.
- Let us consider using a coding scheme which is based on a systematic $[6, 3]$-MDS code.
- $[6, 3]$-MDS code:
  - 3 message symbols are encoded to obtain 6 code symbols (the 3 message symbols $+$ 3 parity symbols),
  - any 3 out of 6 code symbols can give the 3 message symbols.



An MDS-based Scheme for Burst Erasures



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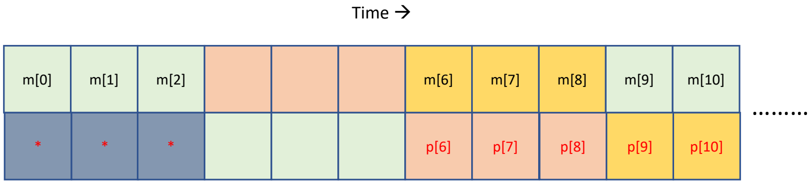

Packets **x**[3], **x**[4], **x[5]** erased (burst erasure of length 3)

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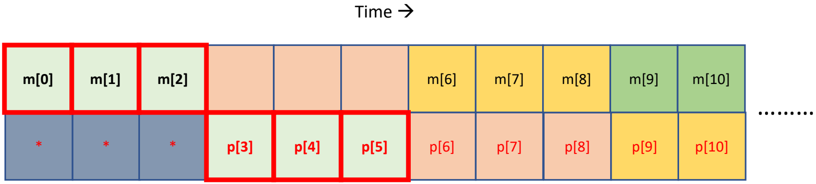

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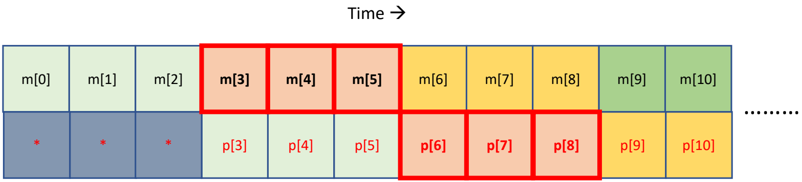
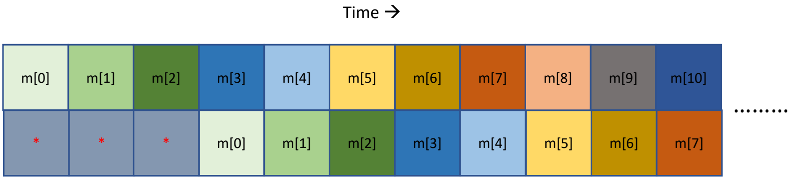


worst-case delay is 5 time units!

A Delayed Repetition Code



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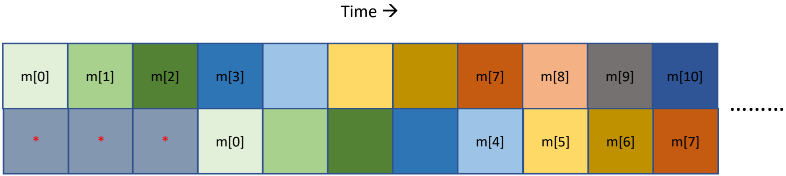

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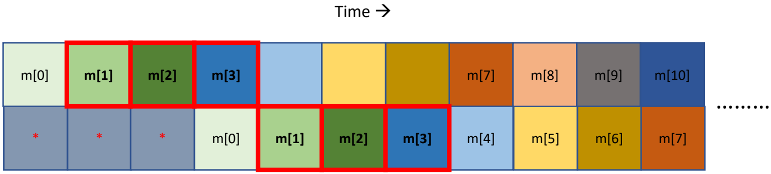

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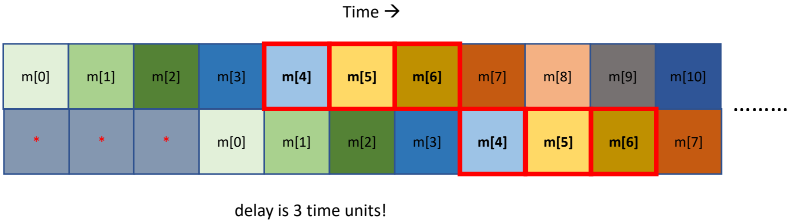
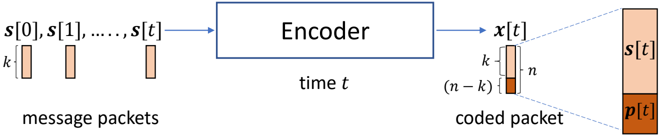


MDS Code vs Delayed Repetition Code

- Both codes have the same rate: $\frac{1}{2}$.
- MDS code is inferior with respect to the delay performance.
- However, we didn't consider arbitrary erasure patterns in the example.
- In this talk, we discuss "optimal" codes that can tolerate arbitrary erasures and burst erasures.

Codes for Streaming: Setting

- Introduced by Martinian and Sundberg [1].
- Message packets: $\mathbf{s}[0], \mathbf{s}[1], \ldots$ to be encoded and sent over the channel.

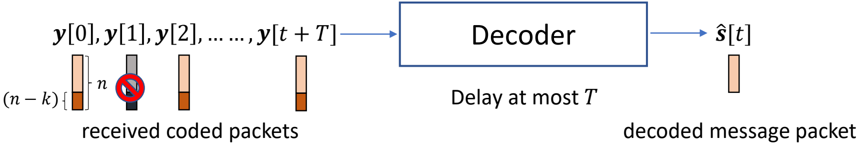


- We consider a systematic, causal encoder, i.e., $\mathbf{x}[t] = [\mathbf{s}[t]; \mathbf{p}[t]]$.
- $\mathbf{p}[t] \in \mathbb{F}_q^{n-k}$ is a function of message packets $\{\mathbf{s}[0], \mathbf{s}[1], \ldots, \mathbf{s}[t]\}$.
- Rate is naturally defined as $\frac{k}{n}$.

[1] E. Martinian and C. W. Sundberg, "Burst erasure correction codes with low decoding delay," *IEEE Trans. Inf. Theory*, 2004.

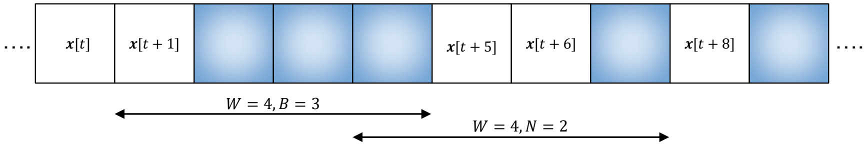
Codes for Streaming: Setting

- Channel introduces erasures at (coded) packet level.
- Decoder with delay constraint T.
- Decoded message packet, $\hat{\mathbf{s}}[t]$ is a function of coded packets $\{\mathbf{y}[0], \mathbf{y}[1], \ldots, \mathbf{y}[t+T]\}$.



Sliding-Window Channel Model: $\mathcal{S}(N, B, W)$

- Consider any sliding window of width W.
- The channel [2] can only introduce one of the following 2 erasure patterns:
  1. $\leq N$ erasures at arbitrary locations within sliding-window of width W,
  2. an erasure-burst of length at most B, $B \geq N$.
- Decoding delay constraint, T, $B \leq T$
- Example channel realization with $N = 2, B = 3, W = 4$:



[2] A. Badr, P. Patil, A. Khisti, W. Tan, and J. G. Apostolopoulos, "Layered Constructions for Low-Delay Streaming Codes," *IEEE Trans. Inf. Theory*, 2017.

Streaming Capacity

- A rate R is achievable over $\mathcal{S}(N, B, W)$ with decoding delay constraint $T \implies$ there exists a streaming code with rate R that can recover from all the erasure patterns permitted by $\mathcal{S}(N, B, W)$ with decoding delay $\leq T$ for each message packet.
- The following upper bound for an achievable rate R is known [2]:

$$R \leq \frac{T_{\text{eff}} - N + 1}{B + T_{\text{eff}} - N + 1}, \tag{1}$$

where $T_{\text{eff}} \triangleq \min\{T, W - 1\}$.

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Streaming Capacity

- Achievability of the bound not known in general.
- In this work, we provide codes that meet the bound for all feasible parameters of N, B, W, T.

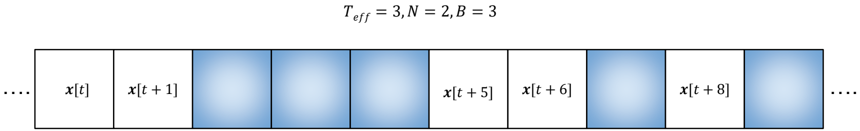
- A concurrent work by Fong et al. [3] also proves the achievability of the rate upper-bound.

[3] S. L. Fong, A. Khisti, B. Li, W. Tan, X. Zhu, and J. Apostolopoulos, "Optimal Streaming Codes for Channels with Burst and Arbitrary Erasures," in *Proc. ISIT*, 2018.

A Simplified Code-Design Criteria

- Consider a streaming code $\mathcal{C}_{\text{str}}$ which permits recovery of any packet $\mathbf{x}[t]$ with a delay of $T_{\text{eff}} \triangleq \min\{T, W-1\}$ even in presence of
  - a burst erasure of length at most B, or
  - at most N isolated erasures.

  This design constraint is
  - stricter in terms of delay constraint,
  - seems to deal with a more relaxed erasure model (only one burst of length B over all time or only N random erasures over all time)
    - but can actually handle a burst of length B or N random erasures in any window of width W
- We illustrate with an example: let $W = 4, T = 4 \implies T_{eff} = 3; N = 2, B = 3.$



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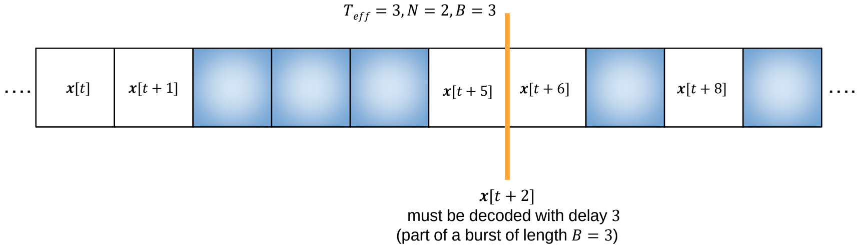

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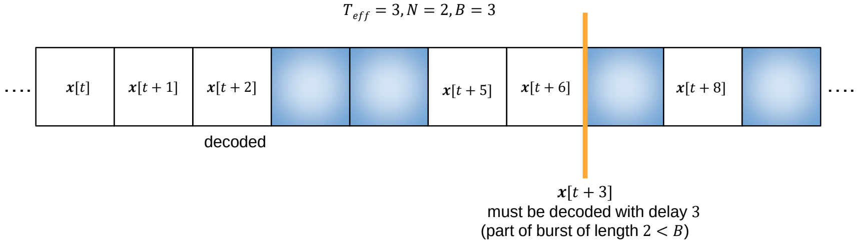

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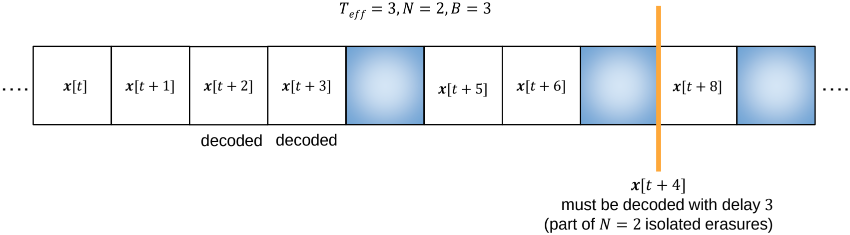


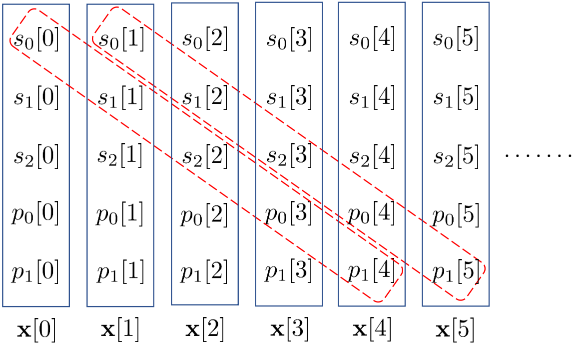
The Simplified Code-Design Criteria is Sufficient

Hence, to design a streaming code for $\mathcal{S}(N, B, W)$ with a delay-constraint T, it suffices to design a streaming code $\mathcal{C}_{\text{str}}$ which permits recovery of packet $\mathbf{x}[t]$ with delay of $T_{\text{eff}} \triangleq \min\{T, W-1\}$ even in the presence of

- a burst erasure of length at most B, or
- at most N isolated erasures.

Diagonal Coding

- A known technique that enables one to convert a *canonical* $[n, k]$ block code $\mathcal{C}$ into a convolutional code.
- Example: Let the canonical code $\mathcal{C}$ be a systematic $[5, 3]$ code. Use $\mathcal{C}$ to code across diagonals:



- Rate of the convolutional code remains as $\frac{k}{n}$.

Desired Properties on the Canonical Block Code $\mathcal{C}$

- We intend to code across diagonals using the canonical code to obtain the desired streaming code $\mathcal{C}_{\text{str}}$ for the $\mathcal{S}(N, B, W)$ channel.
- Requirements on canonical $[n, k]$ linear block code $\mathcal{C}$:

- $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, $\Delta_i \triangleq \min\{i + T_{\text{eff}}, n-1\}$.
- Let c_i be recoverable from $\{c_j : j \in [0, \Delta_i]\}$ even in presence of one of the following:
  - $\leq N$ arbitrary erasures among the symbols $\{c_j : j \in [0, \Delta_i]\}$,
  - a burst of length $\leq B$ affecting the symbols $\{c_j : j \in [0, \Delta_i]\}$.

- Goal: Construct a canonical code $\mathcal{C}$ with dimension $k = (T_{\text{eff}} - N + 1)$ and code-length $n = (B + T_{\text{eff}} - N + 1)$, which will then give rise to an $\mathcal{C}_{\text{str}}$ that meets the rate upper-bound (1).

An Example Canonical Code $\mathcal{C}$ with the Desired Properties

- Consider a $[5, 2]$ linear block code $\mathcal{C}$, with parameters $N = 2, B = 3, W = 4, T_{eff} = 3$.

$T_{eff} = 3, N = 2, B = 3$

c_0	c_1	c_2	c_3	c_4

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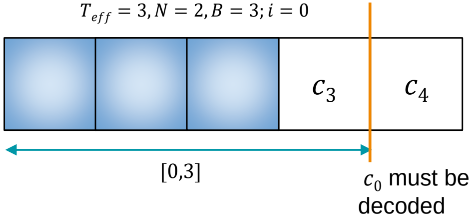

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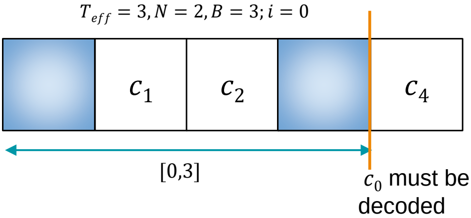

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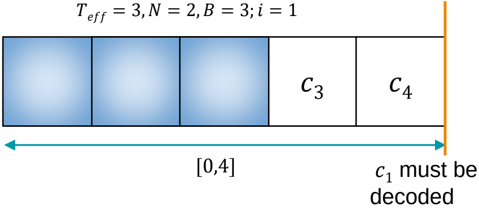

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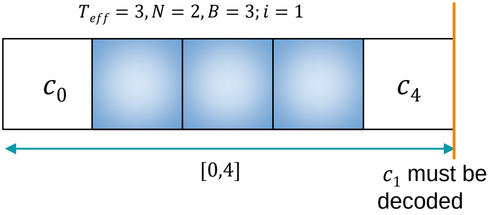

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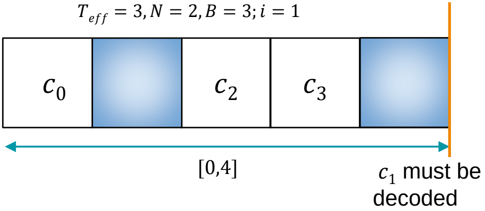

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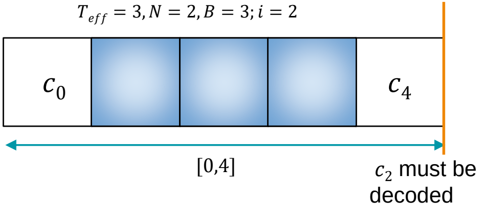

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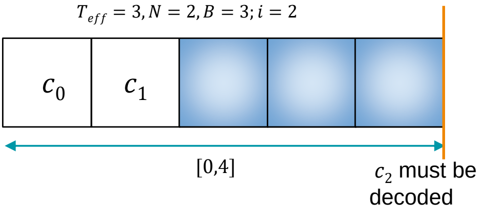

Streaming Code from $\mathcal{C}$ using Diagonal Coding

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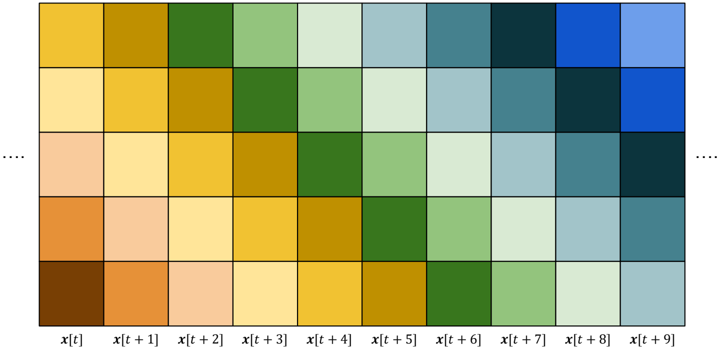

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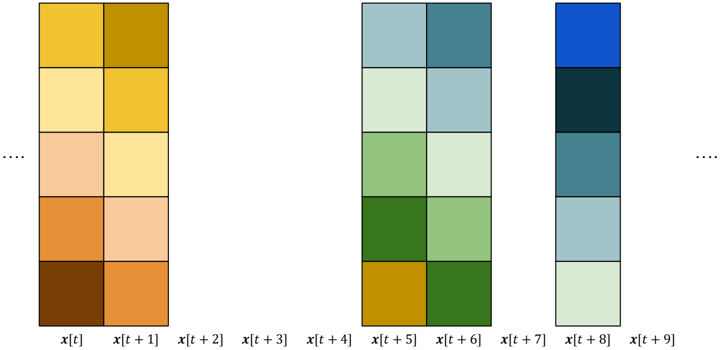

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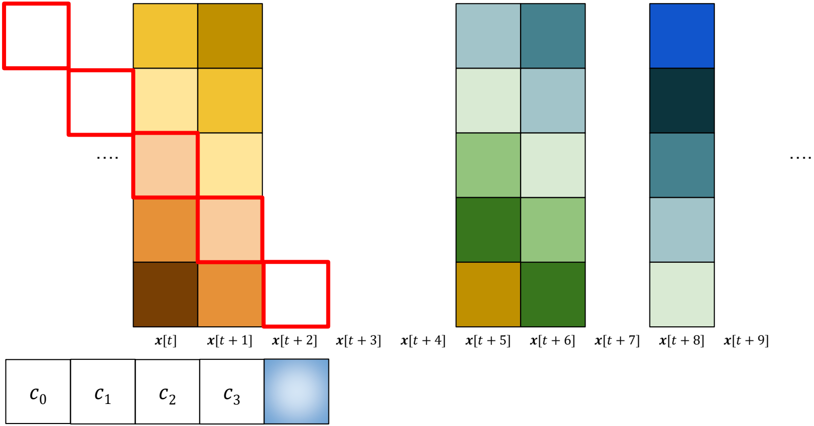

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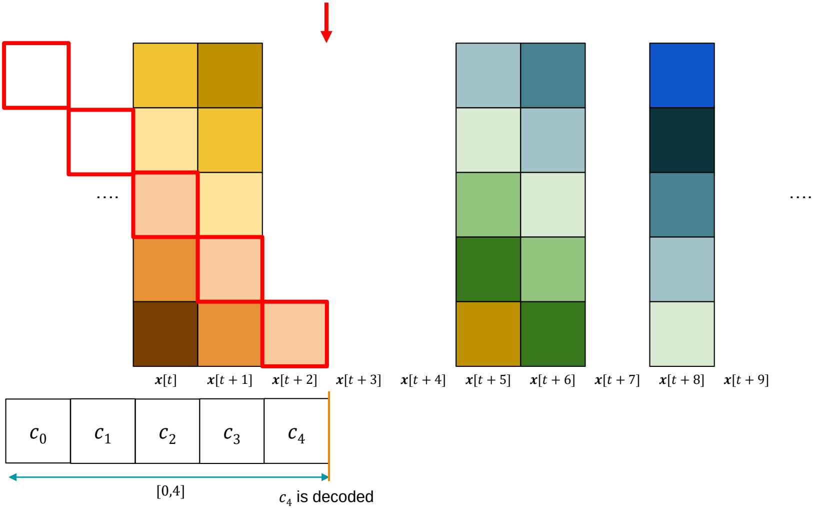

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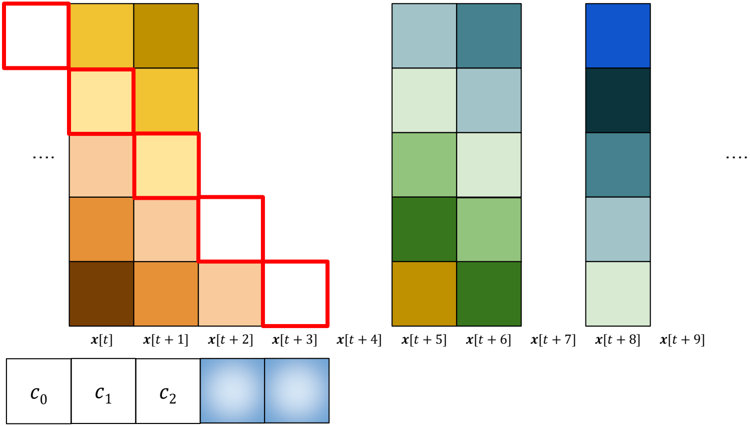

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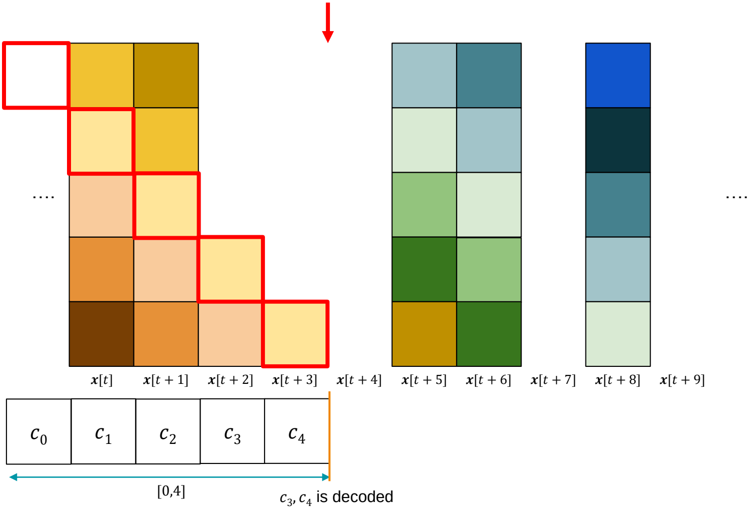

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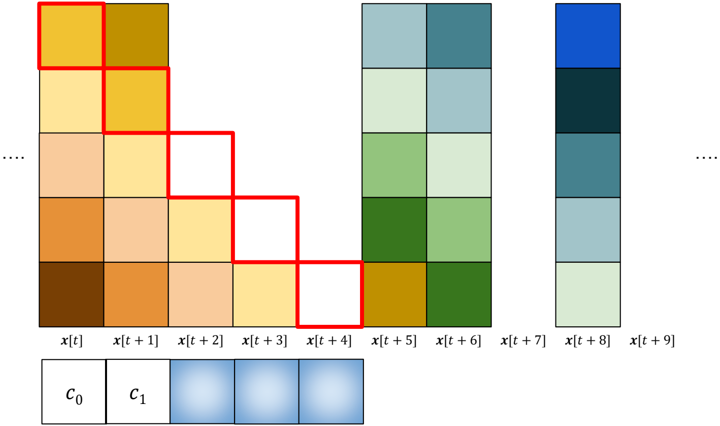

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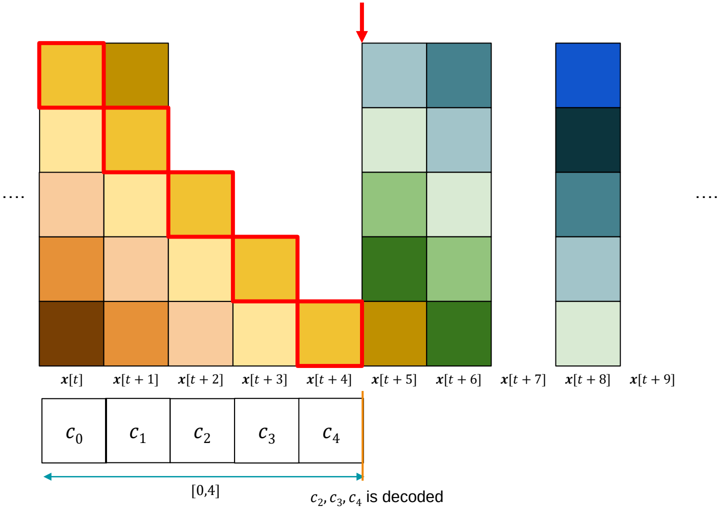

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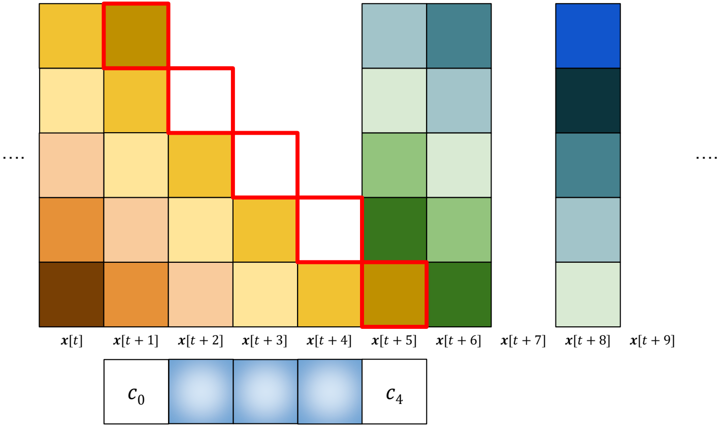

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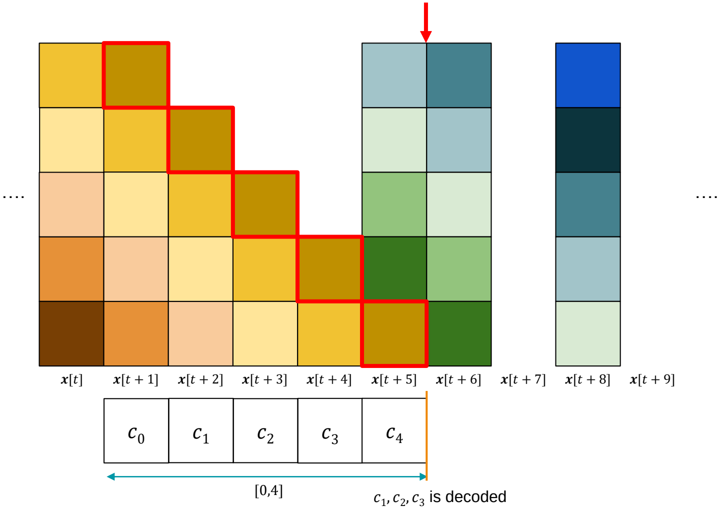

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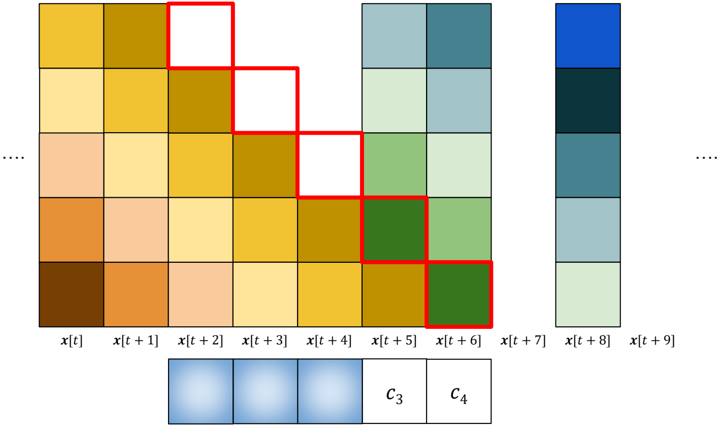

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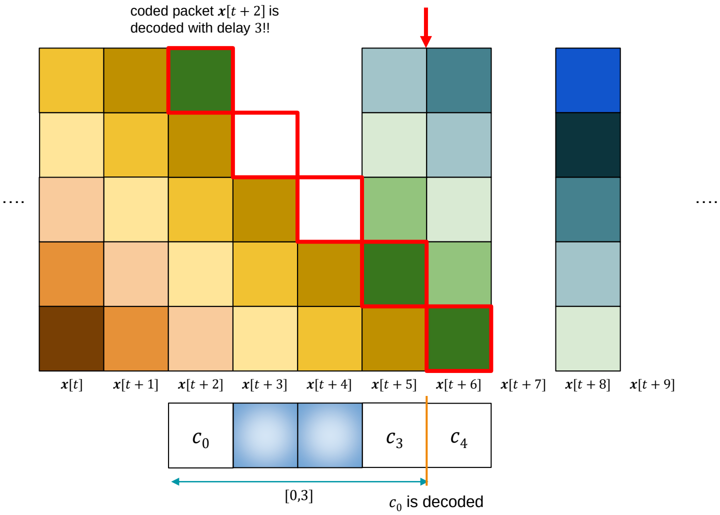

Designing $\mathcal{C}$: Construction-A

- For given parameters N, B, W, T_{eff}: $R \leq \frac{T_{eff}-N+1}{B+T_{eff}-N+1}$.
- We shall construct a rate-optimal code $\mathcal{C}$ with $k = (T_{eff} - N + 1)$, $n = (B + T_{eff} - N + 1)$.
- For $\mathbf{c} \in \mathcal{C}$, c_i can be recovered from coordinates $[0, \min\{i + T_{eff}, n-1\}]$ even in presence of length-B burst or N isolated erasures.
- Let $T = aB + \delta$, where $a \geq 0$ and $1 \leq \delta \leq B$.
- Constraint: $\delta \geq (B - N)$ (can be removed)

- Basic idea: Use an N-erasure correcting $[k + N, k]$ MDS code + $(B - N)$ low-weight check-sums.

Designing $\mathcal{C}$: Construction-A

- Consider a systematic $[n_{\text{MDS}} = T+1, k_{\text{MDS}} = k, d_{\min} = N+1]$ MDS code $\mathcal{C}_{\text{MDS}}$ over $\mathbb{F}_q \subseteq \mathbb{F}_{q^2}$.
- Let $\mathbf{c} \triangleq (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$.
- $(c_0, c_1, c_2, \ldots, c_{T-1}, c_{n-1}) \in \mathcal{C}_{\text{MDS}}$.
- For $0 \leq j \leq (N - T - 2)\}$, $c_{T+j} = \alpha c_j + c_{j+B} + \ldots + c_{j+aB}$, where $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.
- Field-size requirement $O(T^2)$.

Example: Construction-$\mathcal{A}$

- $N = 2, B = 4, T_{\text{eff}} = 10, k = 9, n = 13.$
- $[11, 9, 3]$ MDS code: $\mathcal{C}_{\text{MDS}}$.
- $(c_0, c_1, \ldots, c_9, c_{12}) \in \mathcal{C}_{\text{MDS}}$

c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	$\alpha c_0 + c_4 + c_8$	$\alpha c_1 + c_5 + c_9$	c_{12}
										c_{10}	c_{11}	

Example: Construction-A: Burst Erasure Correction

- $N = 2, B = 4, T_{\text{eff}} = 10, k = 9, n = 13.$
- $[11, 9, 3]$ MDS code: $\mathcal{C}_{\text{MDS}}$.
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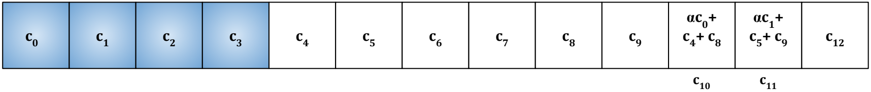

Example: Construction-A: Burst Erasure Correction

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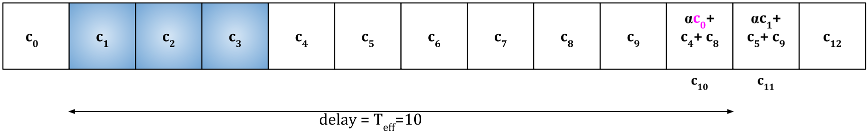

Example: Construction-A: Burst Erasure Correction

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Example: Construction-A: Burst Erasure Correction

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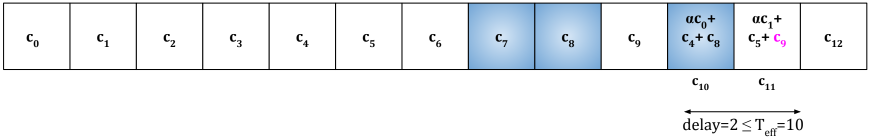
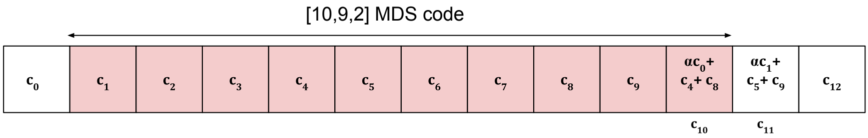


Example: Construction-A: Isolated Erasures

- $N = 2, B = 4, T_{\text{eff}} = 10, k = 9, n = 13$.
- $[11, 9, 3]$ MDS code: $\mathcal{C}_{\text{MDS}}$.
- $(c_0, c_1, \ldots, c_9, c_{12}) \in \mathcal{C}_{\text{MDS}}$.
- $\mathcal{C}$ can recover from any N erasures because of the embedded MDS code $\mathcal{C}_{\text{MDS}}$.
- Delay constraints need to be checked for isolated erasures involving at least one among the coordinates $\{c_0, c_1\}$.



Example: Construction-A: Isolated Erasures

- $N = 2, B = 4, T_{\text{eff}} = 10, k = 9, n = 13.$
- $[11, 9, 3]$ MDS code: $\mathcal{C}_{\text{MDS}}$.
- $(c_0, c_1, \dots, c_9, c_{12}) \in \mathcal{C}_{\text{MDS}}$.
- $\mathcal{C}$ can recover from any N erasures because of the embedded MDS code $\mathcal{C}_{\text{MDS}}$.
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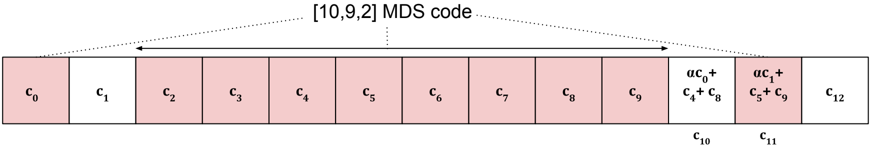

Remark

- There is a Construction-B which removes the constraint $\delta \geq (B - N)$.
- Uses linearized polynomials and require a field-size, which is exponential in T.

Thank You!