Index Coding using Interference Management

Niranjana Ambadi

Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore - India

ECE Students Seminar Series, October 2017.

Index Coding Problem

Single sender, Multiple users

- Single source transmitting messages from a finite alphabet $\mathcal{M} = \{x_1, x_2, \ldots, x_M\}$ to K receivers/destinations
- A receiver $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$ wants $\mathcal{W}_i \subseteq \mathcal{M}$ and knows $\mathcal{A}_i \subseteq \mathcal{M}$ a priori as side-information
- Noiseless Index Coding Problem: Identify the minimum number of transmissions (optimal length) so that all the receivers can decode their wanted messages using the transmitted symbols and their prior information.[1]

[1] Z. Bar-Yossef, Z. Birk, T. S. Jayram and T. Kol, "Index coding with side information", in *Proc. 47th Annu. IEEE Symp. Found. Comput. Sci.*, Oct. 2006, pp. 197-206.

Topological Interference Management

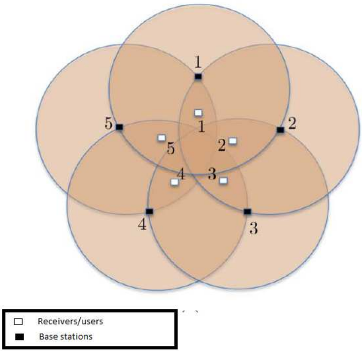

Figure : TIM setting- example

Problem Formulation

Single Unicast Neighboring Interference Symmetric Index Coding

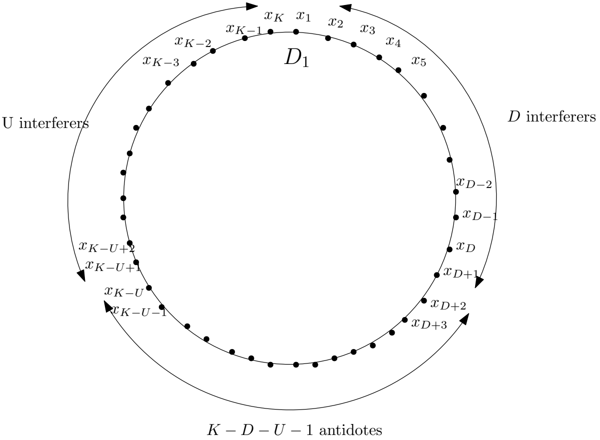

Figure : Interferers and antidotes at destination D_1

Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

Demand Set, $\mathcal{W}_i$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Side-information, $\mathcal{A}_i$	x_4, x_5	x_5, x_6	x_6, x_7	x_7, x_1	x_1, x_2	x_2, x_3	x_3, x_4

- Optimal length (minrank): ???

Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

Demand Set, $\mathcal{W}_i$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Side-information, $\mathcal{A}_i$	x_4, x_5	x_5, x_6	x_6, x_7	x_7, x_1	x_1, x_2	x_2, x_3	x_3, x_4

- Optimal length (minrank): ??? ,
- Index code:???

Definitions

Interferers

Interferers: For each destination $D_i \in \mathcal{D}$ the set of interfering messages is given by $\mathcal{I}_i = (\mathcal{W}_i \cup \mathcal{A}_i)^c$.

Demand Set, $\mathcal{W}_i$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Side-information, $\mathcal{A}_i$	x_4, x_5	x_5, x_6	x_6, x_7	x_7, x_1	x_1, x_2	x_2, x_3	x_3, x_4
Interferers, $\mathcal{I}_i$	x_2, x_3 x_6, x_7	x_3, x_4 x_7, x_1	x_4, x_5 x_4, x_5	x_5, x_6 x_2, x_3	x_6, x_7 x_3, x_4	x_7, x_1 x_4, x_5	x_1, x_2 x_5, x_6

Definitions

Scalar Linear Codes

Scalar Linear Index Code: When $\mathcal{S}$ is a finite field, an $(\mathcal{S}, n, \mathcal{R})$ index code is scalar linear if, for the source with M messages, $\mathcal{M} = \{x_1, x_2, \ldots, x_M\}$, the transmitted symbol sequence is given by,

$$\mathbb{S}^n = \sum_{j=1}^{M} V_j x_j.$$

The $n \times 1$ vector V_j- the precoding vector (or beamforming vector) for the message x_j.

$$\mathbb{S}^n = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_6 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_7$$

Definitions

Symmetric Index Coding

Symmetric Index Coding: An index coding problem is symmetric if for any two receivers D_i and D_j, $i, j \in \lfloor K \rceil; j \neq i$ there exists

1. a bijection $\pi : \mathcal{A}_i \rightarrow \mathcal{A}_j$ such that $\pi(x_k) = x_{k+j-i}$; and
2. a bijection $\omega : \mathcal{W}_i \rightarrow \mathcal{W}_j$ such that $\omega(x_k) = x_{k+j-i}$.

In simple Terms!!! Relative to its index, each destination has identical sets of wanted messages and side-information

Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

Demand Set, $\mathcal{W}_i$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Side-information, $\mathcal{A}_i$	x_4, x_5	x_5, x_6	x_6, x_7	x_7, x_1	x_1, x_2	x_2, x_3	x_3, x_4

Sample Problem

Index Coding Problem with $K = 7$ destinations. $D_i \triangleq (\mathcal{W}_i, \mathcal{A}_i)$.

Demand Set, $\mathcal{W}_i$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Side-information, $\mathcal{A}_i$	x_4, x_5	x_5, x_6	x_6, x_7	x_7, x_1	x_1, x_2	x_2, x_3	x_3, x_4

- Optimal length (minrank)$= 4$,
- Index code: $x_7 + x_4,\ x_1 + x_5,\ x_2 + x_6,\ x_3$.

Algorithm

- Use interference alignment.
- The messages in a given demand set $\mathcal{W}_i$ mustbe sent independently
- The messages interfering at a given D_i that come as part of the demand set of another reciver must be sent independently to each other as well as to the messages in the demand set.

The strategy is...:

Algorithm

- Use interference alignment.
- The messages in a given demand set $\mathcal{W}_i$ mustbe sent independently
- The messages interfering at a given D_i that come as part of the demand set of another reciver must be sent independently to each other as well as to the messages in the demand set.

The strategy is...:

"...Count Dimensions needed to avoid Interference."

Finding the optimal length, λ

Demonstrating the algorithm through example

1. Finding the interference pattern at each destination: Pick up any receiver, say D_1. Find which receiver shares the maximum number of interferers with D_1. Answer: D_2.
   Next find which receiver shares the maximum interferers with both D_1 and D_2. Answer: D_3
   Whenever $D \geq U$, we can conclude that the receiver D_{j+1} will have the maximum number of interferers in common with $D_1, D_2, \ldots, D_{j-1}$ for $j < U$.

$\mathcal{W}_1$	$\mathcal{A}_1$	$\mathcal{I}_1$	$\{\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_{U+1}\}$
$\{x_1\}$	$\{x_4, x_5\}$	$\{x_2, x_3, x_6, x_7\}$	$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$

Finding the optimal length, λ

2. In general, when $D > U$, take
$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_U \cap \mathcal{I}_{a_1} \cap \cdots \cap \mathcal{I}_{a_{i-1}} \cap \mathcal{I}_j$, where
$a_i \triangleq arg\{\max_j |\mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_{a_{i-1}} \cap \mathcal{I}_j|; x_j \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_{a_{i-1}}\}$.
3. As there are only a finite number of receivers, the intersection
$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_{a_1} \cap \cdots \cap \mathcal{I}_{a_n} \to \phi$ for some destination, say, D_{a_n}.
We compute z as follows:

$$\begin{aligned} z &= |\mathcal{W}_1| + |\mathcal{I}_1 \cap \mathcal{W}_2| + |\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{W}_3| + \cdots \\ &+ |\mathcal{I}_1 \cap \cdots \cap \mathcal{W}_U| + |\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_U \cap \mathcal{W}_{a_1}| + \\ &+ |\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_U \cap \mathcal{I}_{a_1} \cap \cdots \cap \mathcal{I}_{a_{n-1}} \cap \mathcal{W}_{a_n}|. \end{aligned}$$

For the example, $z = |\mathcal{W}_1| + |\mathcal{I}_1 \cap \mathcal{W}_1| + |\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{W}_3| = 3$

2. We define S_1' as the set of messages that were taken into account at D_1 while computing z as

$$S_1' = \{x_1, x_2, \ldots, x_U, x_{a_1}, x_{a_2}, \ldots, x_{a_n}\}. \quad (1)$$

Due to the symmetry of the problem, interference pattern is the same at all the receivers.Hence, at D_i,

$$S_i' = \{x_i, x_{2+i-1}, \ldots, x_{a_1+i-1}, \ldots, x_{a_n+i-1}\}. \quad (2)$$

Finding the optimal length, λ

3. At each destination a set of $z = |S_1'|$ messages with consecutive indices must have linearly independent pre-coding vectors.
   This is possible only if $K = nz$.
   If $K \neq nz$, λ is defined as $z + 1$, so that the pre-coding vectors of all consecutive sets of z messages can be chosen to be linearly independent. Thus we have,

$$\lambda \triangleq \begin{cases} z+1, & \text{if} \quad K \neq nz. \\ z, & \text{if} \quad K = nz. \end{cases} \tag{3}$$

For the example: $z = 3, K = 7 \neq nz, \ \lambda = z + 1 = 4$

Constructing Optimal Scalar Linear Index Code

for Single Unicast Neighboring Interference Symmetric Index Coding

1. The optimal length λ is found.
2. Let $\mathcal{T} = \{T_1, T_2, \ldots, T_\lambda\}$ be the set of columns of the identity matrix $I_{\lambda\times\lambda}$ over $\mathbb{F}_2$. Every message in $S_1 = \{x_{K-\lambda+D+2}, \ldots, x_K, x_1, x_2, \ldots, x_{D+1}\}$ is assigned a distinct vector in $\mathcal{T}$ as its pre-coding vector.
   $V_{K-\lambda+D+2} = T_1,\ V_{K-\lambda+D+3} = T_2,\ \ldots,\ V_{D+1} = T_\lambda.$
3. Let $K = q(K-U-1) + r$.
   For $i \in \{D+2, D+3, \ldots, D+r+1\}$,

$$V_i \triangleq \begin{aligned} & V_{i+(K-\lambda)} + V_{i+(K-\lambda)+(K-U-1)} + \ldots \\ & + V_{i+K-\lambda+(q-1)(K-U-1)}. \end{aligned} \tag{4}$$

4. For $i \in \{D+r+2, D+r+3, \ldots, K\}$,

$$V_i \triangleq V_{i-(K-\lambda)} + V_{i+(K-\lambda)} + V_{i+(K-\lambda)+(K-U-1)} + \cdots + V_{i+(K-\lambda)+(q-2)(K-U-1)}. \quad (5)$$

5. The set of λ transmitted symbols of the $(\mathbb{F}_2, \lambda, \mathcal{R})$ index code with $\mathcal{R} = (\frac{1}{\lambda}, \frac{1}{\lambda}, \ldots, \frac{1}{\lambda})$ is given by

$$\mathbb{S}^\lambda = \sum_{i=1}^{K} V_i x_i.$$

Sample Problem

Finding λ and the optimal code

Example

$\mathcal{W}_1$	$\mathcal{A}_1$	$\mathcal{I}_1$	$\{\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_{U+1}\}$	λ_1
$\{x_1\}$	$\{x_4, x_5\}$	$\{x_2, x_3, x_6, x_7\}$	$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$	$z = \|\mathcal{W}_1\| + \|\mathcal{I}_1 \cap \mathcal{W}_2\| + \|\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{W}_3\| = 1 + 1 + 1 = 3$
				$\lambda = z + 1 = 4$

Sample Problem

Finding λ and the optimal code

Example

$\mathcal{W}_1$	$\mathcal{A}_1$	$\mathcal{I}_1$	$\{\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_{U+1}\}$	λ_1
$\{x_1\}$	$\{x_4, x_5\}$	$\{x_2, x_3, x_6, x_7\}$	$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$	$z = \|\mathcal{W}_1\| + \|\mathcal{I}_1 \cap \mathcal{W}_2\| + \|\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{W}_3\| = 1 + 1 + 1 = 3$
				$\lambda = z + 1 = 4$

- Let T_1, T_2, T_3, T_4 be the columns of the 4×4 identity matrix. Choose $V_7 = T_1, V_1 = T_2, V_2 = T_3, V_3 = T_4$.

Sample Problem

Finding λ and the optimal code

Example

$\mathcal{W}_1$	$\mathcal{A}_1$	$\mathcal{I}_1$	$\{\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_{U+1}\}$	λ_1
$\{x_1\}$	$\{x_4, x_5\}$	$\{x_2, x_3, x_6, x_7\}$	$\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \phi$	$z = \lvert\mathcal{W}_1\rvert + \lvert\mathcal{I}_1 \cap \mathcal{W}_2\rvert + \lvert\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{W}_3\rvert = 1 + 1 + 1 = 3$
				$\lambda = z + 1 = 4$

- Let T_1, T_2, T_3, T_4 be the columns of the 4×4 identity matrix. Choose $V_7 = T_1, V_1 = T_2, V_2 = T_3, V_3 = T_4$.

$$\mathbb{S}^{\lambda} = L\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & - & - & - \\ 0 & 1 & 0 & 0 & - & - & - \\ 0 & 0 & 1 & 0 & - & - & - \\ 0 & 0 & 0 & 1 & - & - & - \end{bmatrix} \begin{bmatrix} x_7 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

- Now $K = q(K - U - 1) + r$. Here $7 = 1(7 - 2 - 1) + 3$. Using (4), $V_4 = V_{4+(7-4)} = V_7$, $V_5 = V_{5+(7-4)} = V_1$, $V_6 = V_{6+(7-4)} = V_2$.

$$\mathbb{S}^{\lambda} = \mathbf{L}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_7 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

- Now $K = q(K - U - 1) + r$. Here $7 = 1(7 - 2 - 1) + 3$. Using (4), $V_4 = V_{4+(7-4)} = V_7$, $V_5 = V_{5+(7-4)} = V_1$, $V_6 = V_{6+(7-4)} = V_2$.

$$\mathbb{S}^{\lambda} = \mathbf{L}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_7 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

- The corresponding index code given by

$$x_7 + x_4, \quad x_1 + x_5, \quad x_2 + x_6, \quad x_3.$$

Optimal lengths of consecutive non-neighboring antidotes single unicast symmetric index coding problem

$K=11$		$K=11$		$K=10$		$K=10$	
$U,D=0,8$	$\lambda=9$	$U,D=0,7$	$\lambda=8$	$U,D=0,7$	$\lambda=8$	$U,D=0,6$	$\lambda=7$
$U,D=1,7$	$\lambda=9$	$U,D=1,6$	$\lambda=8$	$U,D=1,6$	$\lambda=8$	$U,D=1,5$	$\lambda=6$
$U,D=2,6$	$\lambda=8$	$U,D=2,5$	$\lambda=8$	$U,D=2,5$	$\lambda=7$	$U,D=2,4$	$\lambda=5$
$U,D=3,5$	$\lambda=9$	$U,D=3,4$	$\lambda=6$	$U,D=3,4$	$\lambda=5$	$U,D=3,3$	$\lambda=5$
$U,D=4,4$	$\lambda=6$						

Neighboring antidotes problem

- The consecutive neighboring antidotes problem setting is as follows: For $i \in \lfloor K \rceil$, at receiver D_i, $\mathcal{W}_i = \{x_i\}$, $\mathcal{A}_i = \{x_{i+1}, x_{i+2}, \ldots, x_{i+d}\}$; $d < K$. It is already known [2] that the capacity of this problem is

$$C = (K - d)^{-1}.$$

- An optimal scalar linear code can be constructed by considering $U = 0$ in the neighboring interference problem. With d antidotes, the number of interferers is given by $D = K - d - 1$. Code construction follows.

[2] H. Maleki, V. Cadambe, and S. Jafar, "Index coding an interference alignment perspective", in IEEE *Trans. Inf. Theory,* vol. 60, no.9, pp.5402-5432, Sep. 2014.

Summary

- Interfernce alignment perspective allows us to solve many complicated index coding problems easily
- Scalar linear capacity in IC scenarios can be obtained by counting the number of dimensions required to avoid interference
- Algorithm to find capacity in any symmetric index coding problem.
- Optimal code construction for Neighboring Interference setting with finite number of users/messages.

Thank You!

Email: ambadi@iisc.ac.in